Q Rings

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We show the existence of new stable ring-like localized scalar field configurations whose stability is due to a combination of topological and nontopological charges. In that sense these defects may be called semitopological. These rings are Noether charged and also carry Noether current (they are superconducting). They involve a single complex scalar field and they appear in theories where a U(1) symmetry remains unbroken. We obtain numerical solutions of the field configuration corresponding to large rings and derive virial theorems demonstrating their stability. We also derive the minimum energy field configurations in 3D and simulate the evolution of a finite size Q ring on a three dimensional lattice thus generalizing our demonstration of stability.

Non-topological solitons (Q balls) have been studied extensively in the literature in one, two and three dimensions [1]. They are localized time dependent field configurations with a rotating internal phase and their stability is due to the conservation of a Noether charge Q [2]. In three dimensions, the only localized, stable configurations of this type have been assumed to be of spherical symmetry hence the name Q balls. The generalization of two dimensional (planar) Q balls to three dimensional Q strings leads to loops which are unstable due to tension. Closed strings of this type are naturally produced during the collisions of spherical Q balls and have been seen to be unstable towards collapse due to their tension [3,4].

There is a simple mechanism however that can stabilize these closed loops. It is based on the introduction of an additional phase on the scalar field that twists by $2\pi N$ as the length of the loop is scanned. This phase introduces additional pressure terms in the energy that can balance the tension and lead to a stabilized configuration, the *Q ring*. This type of pressure is analogous to the pressure of the superconducting string loops [5] (also called 'springs' [6]). In fact it will be shown that Q rings carry both Noether charge and Noether current and in that sense they are also superconducting. However they also differ in many ways from superconducting strings. Q rings do not carry two topological invariants like superconducting strings but only one: the winding N of the phase along the Q ring. Their metastability is due not only to the topological twist conservation but also due to the conservation of the Noether charge as in the case of ordinary Q balls. Due to this combination of topological with non-topological invariants Q rings may be viewed as semitopological defects. In what follows we demonstrate the existence and metastability of Q rings in the context of a simple model. We use the term 'metastability' instead of 'stability' because *finite size* fluctuations can lead to violation of cylindrical symmetry and decay of a Q ring to a Q ball as demonstrated by our numerical simulations.

Consider a complex scalar field whose dynamics is determined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - U(|\Phi|) \tag{1}$$

The model has a global $U(1)$ symmetry and the associated onserved Noether current is

$$J_\mu = Im(\Phi^* \partial_\mu \Phi) \tag{2}$$

with conserved Noether charge $Q = \int d^3x \; J_0$. Provided that the potential of (1) satisfies certain conditions [2,1] the model accepts stable Q ball solutions which are described by the ansatz $\Phi = f(r)e^{i\omega t}$. The energy density of this Q ball configuration is localized and spherically symmetric. The stability is due to the conserved charge Q.

In addition to the Q ball there are other similar stable configurations with cylindrical or planar symmetry but infinite, not localized energy in three dimensions. For example an infinite stable Q string that extends along the z axis is described by the ansatz

$$\Phi = f(\rho)e^{i\omega t} \tag{3}$$

where ρ is the azimouthal radius. This configuration has also been called 'planar' or 'two dimensional' Q ball [1].

The energy of this configuration can be made finite and localized in three dimensions by considering closed Q strings. These configurations which have been shown to be produced during spherical Q ball collisions [3,4] are unstable towards collapse due to their tension. In order to stabilize them we need a pressure term that will balance the effects of tension. This term appears if we substitute the string ansatz (3) by the ansatz of the form

$$\Phi = f(\rho)e^{i\omega t}e^{i\alpha(z)} \tag{4}$$

where $\alpha(z)$ is a phase that varies uniformly along the z axis. This phase introduces a new non-zero J_z component to the conserved current density (2). The corresponding current is of the form

$$I_z = \int dz\, \frac{d\alpha}{dz}\, 2\pi \int d\rho\, \rho\, f^2 \tag{5}$$

Consider now closing the infinite Q string ansatz (4) to a finite (but large) loop of size L. The energy of this configuration may be approximated by

$$\begin{aligned} E &= \frac{Q^2}{4\pi L \int d\rho\, \rho\, f^2} + \pi\, L \int d\rho\, \rho\, f'^2 \\ &+ \frac{(2\pi N)^2 \pi}{L} \int d\rho\, \rho\, f^2 + 2\pi L \int d\rho\, \rho U(f) \\ &\equiv I_1 + I_2 + I_3 + I_4 \end{aligned}$$

where we have assumed $\alpha(z) = \frac{2\pi N}{L} z$ and the terms I_i are all positive. Also Q is the charge conserved in $3D$ defined as

$$Q = \omega 2\pi L \int d\rho\, \rho\, f^2 \tag{6}$$

The winding $2\pi N = \int dz\, \frac{d\alpha}{dz}$ is topologically conserved and therefore the current (5) is very similar to the current of superconducting strings.

After a rescaling $\rho \longrightarrow \sqrt{\lambda_1}\rho$, $z \longrightarrow \lambda_2 z$ the rescaled energy may be written as

$$E = \frac{1}{\lambda_1 \lambda_2} I_1 + \lambda_2 I_2 + \frac{\lambda_1}{\lambda_2} I_3 + \lambda_1 \lambda_2 I_4 \tag{7}$$

Derrick's theorem [7] can be satisfied and collapse in any direction can be evaded due to the time dependence [8,9]. We extremize E with respect to $lambda_1, lambda_2$ and set $lambda_1 = lambda_2 = 1$ to obtain the following virial conditions

$$I_3 + I_4 = I_1 \tag{8}$$
$$I_2 + I_4 = I_1 + I_3 \tag{9}$$

In order to check the validity of these conditions numerically we must first solve the ode which f obeys. This is of the form

$$f'' + \frac{1}{\rho} f' + (\omega^2 - (2\pi N)^2/L^2) f - U'(f) = 0 \tag{10}$$

with boundary conditions $f(\infty) = 0$ and $\frac{df}{d\rho}(0) = 0$. Equation (10) is identical with the corresponding equation for 2D Qballs [9] (see ansatz (3)) with the replacement of ω^2 by

$$\omega^2 - \frac{(2\pi N)^2}{L^2} \equiv \omega'^2 \tag{11}$$

Solutions of (10) for various ω' and $U(f) = \frac{1}{2}f^2 - \frac{1}{3}f^3 + \frac{B}{4}f^4$ with $B = 4/9$ were obtained in Ref. [9]. Now it is easy to see that the first virial condition (8) may be written as

$$\omega'^2 \int d\rho\, \rho\, f^2 = 2 \int d\rho\, \rho U(f) \tag{12}$$

This is exactly the virial theorem for 2D Qballs [9] (infinite Q strings) with $N = 0$ and field ansatz given by (3) with ω replaced by ω'. The validity of such a virial condition has been verified in Ref. [9]. This therefore is an effective verification of our first virial condition (8).

The second virial condition (9) can be written (using the first virial (8)) as

$$2I_3 = I_2 \tag{13}$$

which implies that

$$\frac{2\pi N^2}{L^2} = \frac{\int d\rho\, \rho f'^2}{\int d\rho\, \rho f^2} \tag{14}$$

This can be viewed as a relation determining the value of L required for balancing the tension.

These virial conditions can be used to lead to a determination of the energy as

$$E = 2(I_1 + I_3) \tag{15}$$

In the thin wall limit where $2\pi \int d\rho \rho f^2 = A f_0^2$ (A is the surface of a cross section of the Q ring) this may be written as

$$E \simeq \frac{Q^2}{2LAf_0^2} + \frac{(2\pi N)^2 A f_0^2}{2L} \tag{16}$$

and can be minimized with respect to f_0^2. The value of f_0 that minimizes the energy in the thin wall approximation is

$$f_0 = \sqrt{\frac{Q}{2\pi N A}} \tag{17}$$

Substituting this value back on the expression (16) for the energy we obtain

$$E = \frac{2\pi N Q}{L} \tag{18}$$

This is consistent with the corresponding relation for spherical Q balls which in the thin wall approximation lead to a linear increase of the energy with Q.

The above virial conditions demonstrate the persistance of the Q ring configuration towards shrinking or expansion in the two periodic directions of the Q ring torus for large radius.In order to study Q rings of any size we must perform an energy minimization in 3D and subsequently a numerical simulation of the evolution. This is performed in what follows.

We now specify the form of the potential $U(\Phi)$ in order to study numerically the stability and the evolution of Q rings. We consider the potential

$$U(\phi) = \frac{1}{2}|\Phi|^2 - \frac{1}{3}|\Phi|^3 + \frac{B}{4}|\Phi|^4 \tag{19}$$

The ansatz that captures the above mentioned properties of the Q ring is

$$\Phi = f(\rho, z)\, e^{i[\omega t + N\phi]} \tag{20}$$

where the center of the coordinate system now is in the center of the torus that describes the Q ring and the ansatz is valid for *any* radius of the Q ring.

The energy of this configuration is

$$E = \frac{1}{2}\frac{Q^2}{\int f^2 dV} + \frac{1}{2}\int \left[\left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{N^2 f^2}{\rho^2}\right] dV + \frac{1}{2}\int \left[\left(\frac{\partial f}{\partial z}\right)^2\right] dV + \int U(f)\, dV \tag{21}$$

The field equation for Φ is

$$\ddot{\Phi} - \Delta\Phi + \Phi - |\Phi|\Phi + B|\Phi|^2\Phi = 0 \tag{22}$$

Substituting the ansatz (20) we find that $f(\rho, z)$ should satisfy

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - \frac{N^2 f}{\rho^2} + \frac{\partial^2 f}{\partial z^2} + (\omega^2 - 1)f + f^2 - Bf^3 = 0 \tag{23}$$

In order to solve this equation we minimize the energy (21) at fixed Q using a relaxation algorithm. In the algorithm, we have used the initial ansatz:

$$f(\rho, z) e^{(rho - rho_0) + z^2} \tag{24}$$

where ρ_0 is a fixed initial radius. The energy minimization resulted to a non-trivial configuration $f(\rho, z)$ for a given set of parameters B, N, Q in the expression for the energy. We then used (**??**) to calculate ω and constructed the full Q ring configuration using (20). As a further test to the stability of the solution and the energy minimization algorithm we fed the resulting field configuration as initial condition to a leapfrog algorithm simulation the full field evolution in 3D by solving equation (22) in a 81^3 lattice with reflective boundary conditions where the second spatial derivative is set to zero on the boundary. The relaxed configurations with B=4/9, n=1 and various Q's were evolved for about 50 internal Q ring periods $T = 2\pi/\omega$.

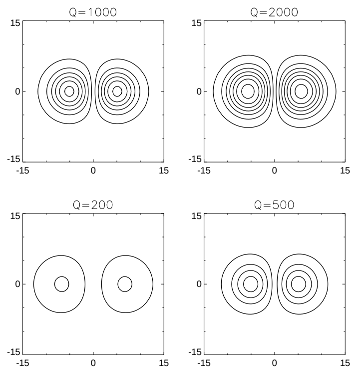

FIG. 1. Charge density contour plots of x-z plane cuts for evolved profiles of relaxed Q rings ($N = 1$). No significant change of the configurations was observed during the evolution of about 50 internal rotation periods.

The contour plots of the final frames of the simulations on the $x - z$ plane are shown in Fig. 1. It was verified that the Q ring configurations evolve with practically no distortion and are metastable despite the long evolution. Subsequent evolution of the configurations shown in Fig. 1 involved non-symmetric fluctuations emerging from the cubic boundaries. These finite size fluctuations were found to lead to a break up and eventual decay of the Q ring to one or more Q balls (depending on the type of fluctuations) after more than 100 internal rotation periods. Thus a Q ring is a metastable as opposed to a stable configuration.

We have repeated the same numerical experiment with $f(\rho, z)$ corresponding to a ring of large radius with

$N = 0$ in the ansatz (20) used as initial condition in the evolution algorithm. The result of the evolution is shown in the frames of Fig. 2 which shows a collapsing Q ring. It collapses due to the lack of pressure support ($N = 0$) within less than 10 internal rotation periods in contrast to the metastable cases of Fig. 1. The unstable collapsing ring produces a pair of Q balls that propagate in opposite directions along the z-axis. This unstable loop is similar to the one previously seen in Q ball simulations of scattering in 3D [3,4].

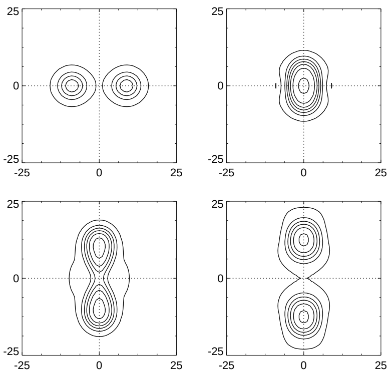

FIG. 2. The evolution of an unstable Q loop with winding $N = 0$ significantly less time than the simulations of Fig.1 (about 10 internal field rotation periods). The ring rapidly collapses and forms a pair of Q balls propagating along the z axis.

The metastable Q ring configuration we have discovered is the simplest metastable ring-like defect known so far. Previous attempts to construct metastable ring-like configurations were based on pure topological arguments (Hopf maps) and required gauge fields to evade Derrick's theorem due to their static nature [10,11]. Metastable moving non-relativistic field configurations have also been constructed [12]. These attempts resulted in complicated models that were difficult to study analytically or even numerically. Q rings require only a single complex scalar field and they appear in all theories that admit stable Q balls including the minimal supersymmetric standard model (MSSM). The simplicity of the theory despite the non-trivial geometry of the field configuration is due to the combination of topological with non-topological charges that combine to secure metastability without added field complications.

The derivation of metastability of this configuration opens up several interesting issues that deserve detailed investigation. Here we outline some of these issues.

- **Formation of Q Rings:** Q rings can form in principle by variations of the Kibble mechanism, the Affleck-Dine mechanism [13] or by collision of Q balls with non-trivial relative phases. The effectiveness of these mechanisms for Q ring formation needs careful numerical and analytical study.
- **Current Quenching:** The increase of the winding N which implies increase of the current along the Q ring can not be arbitrary. In a way similar to superconducting string loops there is a maximum current and winding N_{crit} beyond which the gradient of the phase becomes high enough to favor a zero value of the field inside the ring. The investigation of the dependence of N_{crit} on Q and other parameters of the theory is a interesting issue.
- **Cylindrical Walls:** A stabilizing phase can also be introduced in closed cylindrical Q walls whose tension can thus be balanced by the pressure of the winding phase. What are the properties of these objects which are essentially a simpler version of the Q rings discussed here?

These and other interesting issues emerge as a consequence of our results and await further investigation.

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