CAMB Notes

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Brief summary of some theory relating to CAMB. Includes definitions for the initial power spectra, quintessence equations, series solutions for the regular scalar modes, variable definitions and relationships, and random notes.

I. INITIAL CONDITIONS

For adiabatic modes we choose the initial conditions are set by the amplitude of χ, the comoving curvature perturbation (conserved on super-Hubble scales for the adiabatic mode). In general this is defined via the perturbation in the 3-Ricci scalar, $\eta_a \equiv \frac{1}{2} S D_a \mathcal{R}^{(3)}$ with harmonic coefficient η, on co-moving hypersurfaces (total heat flux $q = 0$, denoted by a bar), so that

$$\bar{\eta} = 2\beta_2 \left[\Phi + \frac{2}{3}\Omega^{-1} \frac{\mathcal{H}^{-1}\Phi' - \Psi}{1+w} \right] \equiv -2\beta_2 \chi.$$

where (for closed models) $\beta_2 = (\nu^2 - 4)/(\nu^2 - 1)$. Here Φ and Ψ are defined as in [1, 2], Ψ is the Newtonian potential, and $\Phi = \Phi_H$ is the potential satisfying the Poisson equation $\beta_2 k^2 \Phi = \frac{1}{2}\kappa S^2 \bar{\mathcal{X}}$. In general $\Phi = -\Psi - S^2 \kappa \Pi / k^2$. In flat models $\beta_2 = \Omega = 1$ and

$$\chi = -\bar{\eta}/2 = -\left[\Phi + \frac{2}{3} \frac{\mathcal{H}^{-1}\Phi' - \Psi}{1+w} \right].$$

For isocurvature modes the initial conditions are set for the corresponding non-zero mode (see later section). We assume statistical isotropy, and the initial power spectrum for the primordial variable X ($= \chi$ for adiabatic) on super-Hubble scales is defined so that

$$\langle |X|^2 \rangle = \sum_\nu \frac{\nu}{\nu^2 + 1} P_X(\nu).$$

Here P_χ gives the power in the 3-Ricci curvature by

$$\langle |\tilde{D}_a \tilde{R}^{(3)}|^2 \rangle = \sum_\nu \frac{\nu}{\nu^2 - 1} \frac{16 k^6}{S^6} \left(\frac{\nu^2 - 4}{\nu^2 - 1} \right)^2 \mathcal{P}_\chi(\nu). \tag{1}$$

During inflation (no anisotropic stress, $\Pi = 0$)

$$\chi = \psi + \frac{\mathcal{H}}{\phi_i'} \delta\phi_i = \Psi + \frac{2}{3}\Omega^{-1} \frac{\mathcal{H}^{-1}\Psi' + \Psi}{1+w}.$$

where ϕ_i is is the inflaton field, and at the end of inflation $\chi = 3/2\Psi$.

In terms of the Weyl tensor variable ϕ (defined in [3], differing by a sign from Φ in [4])

$$\Phi = -\phi - \frac{1}{2}\frac{S^2}{k^2}\kappa\Pi \qquad \Psi = \phi - \frac{1}{2}\frac{S^2}{k^2}\kappa\Pi.$$

In the early radiation dominated era, assuming purely adiabatic perturbations,

$$\chi = \frac{1}{2} \frac{4R_\nu + 15}{R_\nu + 5} \phi = \frac{4R_\nu + 15}{10} \Psi$$

where $R_\nu = \rho_\nu / (\rho_\nu + \rho_\gamma)$ (≈ 0.4). CMBFAST used to set the initial condition $\Psi = -1$, where Ψ is the metric perturbation ψ as defined in [5] in the conformal Newtonian gauge. In CAMB we set $\chi = -1$, same as the new CMBFAST default, so that the CAMB transfer functions are related to those of the old CMBFAST (in flat models) by $(4R_\nu + 15)/10$.

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II. CAMB VARIABLES

CAMB propagates the covariant equations in the zero-acceleration frame, in which the CDM velocity is zero. This is equivalent to the synchronous gauge.

Note that CAMB uses the variable etak $= k\eta_s$, where $\eta_s = -\eta/2$ (in the CDM frame) is the usual synchronous gauge variable. Here η is the curvature perturbation variable as in the previous section. The fractional density perturbation variables are called clxc (Δ_c), clxg (Δ_γ), clxr (Δ_ν), clxb (Δ_b) etc. The synchronous gauge variable h_s appears in the CDM frame as $h'_s = 6h' = 2k\mathcal{Z}$. The $l = 1$ moments are handled in terms of the heat fluxes q_i where $\rho_i q_i = (\rho_i + p_i)v_i$. The total heat flux appears as dgq $= \kappa a^2 \sum_i \rho_i q_i$ in the code, and the total matter perturbation is dgrho $= \kappa a^2 \sum_i \rho_i \Delta_i$.

Other variables are the shear σ and perturbation to the expansion rate Z. They are related by

$$\frac{2}{3}k^2(\beta_2\sigma - Z) = \kappa a^2 \sum_i \rho_i q_i = \text{dgq}$$

$$k^2\eta = \kappa a^2 \sum_i \rho_i \Delta_i - 2k\mathcal{H}Z = \text{dgrho} - 2k\mathcal{H}Z.$$

III. MULTIPOLE EQUATIONS, HARMONIC EXPANSION AND C_l

Here we summarize the equations of the covariant approach.

The photon multipole evolution is governed by the geodesic equation and Thomson scattering, giving [6]

$$\dot{I}_{A_l} + \frac{4}{3}\Theta I_{A_l} + D^b I_{bA_l} - \frac{l}{2l+1}D_{\langle a}I_{A_{l-1}\rangle} + \frac{4}{3}IA_{a_1}\delta_{l1} - \frac{8}{15}I\sigma_{a_1a_2}\delta_{l2} = -n_e\sigma_T\left(I_{A_l} - I\delta_{l0} - \frac{4}{3}Iv_{a_1}\delta_{l1} - \frac{2}{15}\zeta_{a_1a_2}\delta_{l2}\right) \quad (2)$$

where I_{A_l} is taken to be zero for $l < 0$ and

$$\zeta_{ab} \equiv \frac{3}{4}I_{ab} + \frac{9}{2}\mathcal{E}_{ab} \quad (3)$$

is a source from the anisotropic stress and E-polarization. The equation for the density perturbation $D_a I$ is obtained by taking the spatial derivative of the above equation for $l = 0$. The corresponding evolution equations for the polarization multipole tensors are [6]

$$\begin{aligned}
\dot{\mathcal{E}}_{A_l} + \frac{4}{3}\Theta\mathcal{E}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2}\mathrm{D}^b\mathcal{E}_{bA_l} - \frac{l}{2l+1}D_{\langle a_l}\mathcal{E}_{A_{l-1}\rangle} - \frac{2}{l+1}\,\mathrm{curl}\,\mathcal{B}_{A_l} &= -n_e\sigma_T(\mathcal{E}_{A_l} - \frac{2}{15}\zeta_{a_1a_2}\delta_{l2}) \\
\dot{\mathcal{B}}_{A_l} + \frac{4}{3}\Theta\mathcal{B}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2}\mathrm{D}^b\mathcal{B}_{bA_l} - \frac{l}{2l+1}D_{\langle a_l}\mathcal{B}_{A_{l-1}\rangle} + \frac{2}{l+1}\,\mathrm{curl}\,\mathcal{E}_{A_l} &= 0. \quad (4)
\end{aligned}$$

For numerical solution we expand the covariant equations into scalar, vector and tensor harmonics. The resulting equations for the modes at each wavenumber can be studied easily and also integrated numerically.

A. Scalar, vector, tensor decomposition

It is useful to do a decomposition into m-type tensors, scalar ($m = 0$), vector ($m = 1$) and 2-tensor ($m = 2$) modes. They describe respectively density perturbations, vorticity and gravitational waves. In general a rank$-\ell$ PSTF tensor X_{A_l} can be written as a sum of $m-$type tensors

$$X_{A_l} = \sum_{m=0}^{l} X_{A_l}^{(m)}. \quad (5)$$

Each $X^{(m)}_{A_l}$ can be written in terms of $l-m$ derivatives of a transverse tensor

$$X^{(m)}_{A_l} = D_{\langle A_{l-m}}\Sigma_{A_m\rangle} \tag{6}$$

where $\mathrm{D}_{A_l} \equiv \mathrm{D}_{a_1}\mathrm{D}_{a_2}\ldots\mathrm{D}_{a_l}$ and Σ_{A_m} is first order, PSTF and transverse $\mathrm{D}^{a_m}\Sigma^m_{A_{m-1}a_m} = 0$. The 'scalar' component is $X^{(0)}$, the 'vector' component is $X^{(1)}_a$, etc. Since GR gives no sources for X_{A_m} with $m > 2$ usually only scalars, vectors and (2-)tensors are considered. At linear order they evolve independently.

B. Harmonic expansion

For numerical work we perform a harmonic expansion in terms of zero order eigenfunctions of the Laplacian $Q^m_{A_m}$,

$$\mathrm{D}^2 Q^m_{A_m} = \frac{k^2}{S^2}Q^m_{A_m}, \tag{7}$$

where $Q^m_{A_m}$ is transverse on all its indices, $\mathrm{D}^{a_m}Q^m_{A_{m-1}a_m} = 0$. So a scalar would be expanded in terms of Q^0, vectors in terms of Q^1_a, etc. We usually suppress the labelling of the different harmonics with the same eigenvalue, but when a function depends only on the eigenvalue we write the argument explicitly, e.g. $f(k)$.

For $m > 0$ there are eigenfunctions with positive and negative parity, which we can write explicitly as $Q^{m\pm}_{A_m}$ when required. Since

$$\mathrm{D}^2(\,\mathrm{curl}\,Q_{A_m}) = \,\mathrm{curl}\,(\mathrm{D}^2 Q_{A_m}) = \frac{k^2}{S^2}\,\mathrm{curl}\,Q_{A_m} \tag{8}$$

they are related by the curl operation. Using the result

$$\mathrm{curl}\;\mathrm{curl}\,Q^m_{A_m} = \frac{k^2}{S^2}\left[1+(m+1)\frac{K}{k^2}\right]Q^m_{A_m} \tag{9}$$

we can choose to normalize the $\pm$ harmonics the same way so that

$$\mathrm{curl}\,Q^{m\pm}_{A_m} = \frac{k}{S}\sqrt{1+(m+1)\frac{K}{k^2}}\,Q^{m\mp}_{A_m}. \tag{10}$$

A rank-ℓ PSTF tensor of either parity may be constructed from $Q^{m\pm}_{A_m}$ as

$$Q^m_{A_l} \equiv \left(\frac{S}{k}\right)^{l-m}\mathrm{D}_{\langle A_{l-m}}Q^m_{A_m\rangle} \tag{11}$$

and a $X^{(m)}_{A_l}$ component of X_{A_l} may be expanded in terms of these tensors. They satisfy

$$\begin{aligned}
\mathrm{D}^2 Q^m_{A_l} &= \frac{k^2}{S^2}\left(1-[l(l+1)-m(m+1)]\frac{K}{k^2}\right)Q^m_{A_l} \\
\mathrm{D}^{a_l}Q^m_{A_{l-1}a_l} &= \beta^m_l\frac{k}{S}\frac{(l^2-m^2)}{l(2l-1)}Q^m_{A_{l-1}} \\
\mathrm{curl}\,Q^{m\pm}_{A_l} &= \sqrt{\beta^m_0}\frac{m}{l}\frac{k}{S}Q^{m\mp}_{A_l}
\end{aligned} \tag{12}$$

where $\beta^m_l \equiv 1-\left\{l^2-(m+1)\right\}K/k^2$ and $l \geq m$.

Dimensionless harmonic coefficients are defined by

$$\sigma^{(m)}_{ab} = \sum_{k,\pm} \frac{k}{S}\sigma^{(m)\pm} Q^{m\pm}_{ab} \qquad H^{(m)}_{ab} = \sum_{k,\pm} \frac{k^2}{S^2} H^{(m)\pm} Q^{m\pm}_{ab}$$

$$q^{(m)}_{a} = \sum_{k,\pm} q^{(m)\pm} Q^{m\pm}_{a} \qquad E^{(m)}_{ab} = \sum_{k,\pm} \frac{k^2}{S^2} E^{(m)\pm} Q^{m\pm}_{ab}$$

$$\pi^{(m)}_{ab} = \sum_{k,\pm} \Pi^{(m)\pm} Q^{m\pm}_{ab} \qquad I^{(m)}_{A_l} = \rho_\gamma \sum_{k,\pm} I^{(m)\pm}_l Q^{m\pm}_{A_l}$$

$$\Omega_a = \sum_{k,\pm} \frac{k}{S}\Omega^{\pm} Q^{1\pm}_{a} \qquad A^{(m)}_{a} = \sum_{k,\pm} \frac{k}{S} A^{(m)\pm} Q^{m\pm}_{a}$$

$$(D_a X)^{(m)} = \sum_{k,\pm} \frac{k}{S} (\delta X)^{(m)\pm} Q^{m\pm}_a \tag{13}$$

where the k dependence of the harmonic coefficients is suppressed. We also often suppress m and $\pm$ indices for clarity. The other multipoles are expanded in analogy with I_{A_l}. The heat flux for each fluid component is given by $q_i = (\rho_i + p_i)v_i$, where v_i is the velocity, and the total heat flux is given by $q = \sum_i q_i$. In the frame in which $\Omega_a = 0$ gradients are purely scalar $(\delta \bar{X})^{(1)} = 0$.

C. Harmonic multipole equations

Expanded into harmonics, the photon multipole equations (2) become

$$I_l' + \frac{k}{2l+1}\left[\beta^m_{l+1}\frac{(l+1)^2 - m^2}{l+1} I_{l+1} - l I_{l-1}\right] = \\ - S n_e \sigma_T \left(I_l - \delta_{l0} I_0 - \frac{4}{3}\delta_{l1} v - \frac{2}{15}\zeta\delta_{l2}\right) + \frac{8}{15}k\sigma\delta_{l2} - 4h'\delta_{l0} - \frac{4}{3}kA\delta_{l1} \tag{14}$$

where $l \geq m$, $I_0 = \delta\rho_\gamma/\rho_\gamma$, $I_l = 0$ for $l < m$, and m superscripts are implicit. The scalar source is $h' = (\delta S/S)'$. The equation for the neutrino multipoles (after neutrino decoupling) is the same but without the Thomson scattering terms (for massive neutrinos see Ref. [7]). The polarization multipole equations (4) become

$$\mathcal{E}^{m\pm\prime}_l + k\left[\beta^m_{l+1}\frac{(l+3)(l-1)}{(l+1)^3}\frac{(l+1)^2 - m^2}{(2l+1)}\mathcal{E}^{m\pm}_{l+1} - \frac{l}{2l+1}\mathcal{E}^{m\pm}_{l-1} - \frac{2m}{l(l+1)}\sqrt{\beta^m_0}\mathcal{B}^{m\mp}_l\right] = -Sn_e\sigma_T(\mathcal{E}^{m\pm}_l - \frac{2}{15}\zeta^{m\pm}\delta_{l2})$$

$$\mathcal{B}^{m\pm\prime}_l + k\left[\beta^m_{l+1}\frac{(l+3)(l-1)}{(l+1)^3}\frac{(l+1)^2 - m^2}{(2l+1)}\mathcal{B}^{m\pm}_{l+1} - \frac{l}{2l+1}\mathcal{B}^{m\pm}_{l-1} + \frac{2m}{l(l+1)}\sqrt{\beta^m_0}\mathcal{E}^{m\mp}_l\right] = 0. \tag{15}$$

D. Integral solutions

Solutions to the Boltzmann hierarchies can be found in terms of line of sight integrals. The flat vector and tensor results are given in Ref. [8]. General scalar and tensor results are in [6] (though I_l in that paper differs by a curvature factor).

E. Power spectra

Using the harmonic expansion of I_{A_l} the contribution to the C_l from type-m tensors becomes

$$C^{TT(m)}_l = \frac{\pi}{4}\frac{(2l)!}{(-2)^l (l!)^2}\sum_{k,k',\pm}\langle I^{\pm}_{l,k} I^{\pm}_{l,k'}\rangle Q^{\pm}_{A_l k} Q^{A_l \pm}_{k'}. \tag{16}$$

The multipoles I_l can be related to some primordial variable $X_{A_m} = \sum_k \left(X^+ Q^{m+}_{A_m} + X^- Q^{m-}_{A_m}\right)$ via a transfer function T^X_l defined by $I_l = T^X_l X$. Statistical isotropy and orthogonality of the harmonics implies that

$$\langle X^\pm_k X^\pm_{k'}\rangle = f_X(k)\delta_{kk'} \tag{17}$$

where $\sum_k \delta_{kk'} Y_k = Y_{k'}$ and $f_X(k)$ is some function of the eigenvalue k. The normalization of the $Q^m_{A_l}$ is given by

$$\int dV Q^m_{A_l} Q^{mA_l} = \int dV \left(\frac{-S}{k}\right)^{l-m} Q^m_{A_m} D^{A_{l-m}} Q^m_{A_l} = \alpha^m_l \frac{(-2)^{l-m}(l+m)!(l-m)!}{(2l)!} N \tag{18}$$

where we have integrated by parts repeatedly, then repeatedly applied the identity for the divergence (12). The normalization is $N \equiv \int dV Q^{A_m} Q_{A_m}$ and $\alpha^l_m \equiv \prod_{n=m+1}^{l} \beta^m_n$. By statistical isotropy $C_l = (1/V)\int dV C_l$ and hence

$$C_l^{TT(m)} = \frac{\pi}{4}\frac{(l+m)!(l-m)!}{(-2)^m(l!)^2}\sum_{k,\pm}\frac{N}{V}\alpha^m_l |T^X_l(k)|^2 f_X(k) \tag{19}$$

We choose to define a power spectrum $P_X(k)$ so that the real space isotropic variance is given by

$$\langle |X_{A_m} X^{A_m}|\rangle = \sum_{k,\pm}\frac{|N|}{V} f_X(k) \equiv \int d\ln k\; P_X(k) \tag{20}$$

so the CMB power spectrum becomes

$$C_l^{TT(m)} = \frac{\pi}{4}\frac{(l+m)!(l-m)!}{2^m(l!)^2}\int d\ln k\; P_X(k)\alpha^m_l |T^X_l(k)|^2. \tag{21}$$

For a non-flat universe $\int d\ln k$ is replaced by some some other measure which should be specified when defining the power spectrum. In a closed universe the integral becomes a sum over the discrete modes. Note that we have not had to choose a specific representation of Q_{A_m} or $\sum_k$.

The polarization C_l are obtained similarly [6] and in general we have

$$C_l^{JK(m)} = \frac{\pi}{4}\left[\frac{(l+1)(l+2)}{l(l-1)}\right]^{p/2}\frac{(2l)!}{(-2)^l(l!)^2}\frac{\langle J_{A_l}K^{A_l}\rangle}{\rho_\gamma^2} \tag{22}$$

$$= \frac{\pi}{4}\left[\frac{(l+1)(l+2)}{l(l-1)}\right]^{p/2}\frac{(l+m)!(l-m)!}{2^m(l!)^2}\int d\ln k\; P_X(k)\,\alpha^m_l J^X_l K^X_l \tag{23}$$

where JK is TT ($p=0$), EE or BB ($p=2$) or TE ($p=1$). We have assumed a parity symmetric ensemble, so $C_l^{TB} = C_l^{EB} = 0$.

For tensors we use H_T where $h_{ij} = \sum_{k,\pm} 2H_T Q^2_{ij}$ and h_{ij} is the transverse traceless part of the metric tensor. This introduces an additional factor of 1/4 into the result for the C_l in terms of P_h and $T_l^{H_T}$.

The numerical factors in the hierarchy and C_l equations depend on the choice of normalization for the ℓ and k expansions. Neither $e_{\langle A_l\rangle}$ nor Q_{A_l} are normalized, so there are compensating numerical factors in the expression for the C_l. If desired one can do normalized expansions, corresponding to an ℓ- and k-dependent re-scaling of the I_l and other harmonic coefficients, giving expressions in more manifest agreement with Ref. [9].

IV. QUINTESSENCE

The background field equation for a scalar field ψ can be written

$$\frac{d}{dS}\left(S^2\psi'\right) + \frac{S^3 V_{,\psi}}{\mathcal{H}} = 0.$$

This allows $\psi(S)$ and $\psi'(S)$ to be evaluated, and hence the background evolution since $\mathcal{H}$ relates the time and S derivatives. The density and pressure are

$$S^2\rho_\psi = \frac{1}{2}\psi'^2 + S^2 V \qquad S^2 p_\psi = \frac{1}{2}\psi'^2 - S^2 V$$

The linearized exact field equation is

$$\psi'' + 2\mathcal{H}\psi' + D^a D_a\psi = -S^2 V_{,\psi}.$$

where D_a is the spatial covariant derivative. Defining the first order perturbation

$$\mathcal{V}_a \equiv S D_a \psi$$

and fixing to the zero-acceleration frame (the CDM frame) this implies the following equation for the perturbations

$$\mathcal{V}_a'' + 2\mathcal{H}\mathcal{V}_a' + S\mathcal{Z}_a\psi' + S^2 D_a D^b \mathcal{V}_b = -S^2 \mathcal{V}_a V_{,\psi\psi}.$$

This corrects the equation in [3], where other definitions are given. Performing the harmonic expansion one gets the scalar equation

$$\mathcal{V}'' + 2\mathcal{H}\mathcal{V}' + k\psi'\mathcal{Z} + k^2\mathcal{V} = -S^2\mathcal{V}V_{,\psi\psi}.$$

This is what tells you that you can't consistently have $\mathcal{V} = 0$ in an evolving background if there are other perturbations present ($\mathcal{Z} \neq 0$). The heat flux and density perturbation $\mathcal{X}_a^{(\psi)} = SD_a\rho_\psi$ give the scalar relations

$$S^2 q = k\psi'\mathcal{V} \qquad S^2\mathcal{X}_\psi = \psi'\mathcal{V}' + S^2\mathcal{V}V_{,\psi}.$$

There is no contribution to the anisotropic stress. CMBFAST/CAMB propagate quantities like $\kappa S^2\rho$, hence it is useful to work with $\phi \equiv \sqrt{\kappa}\psi$ and similarly for $\mathcal{V}$.

Constant w

A useful parameterization is $w \equiv p_\psi/\rho_\psi =$ constant, which implies

$$\rho_\psi = \rho_0 S^{-3(1+w)} \qquad V = \frac{1}{2}\dot{\psi}^2\frac{1-w}{1+w}.$$

This potential and ψ' are easily obtained in terms of S

$$S^2 V = \frac{1-w}{2}S^2\rho_\psi \qquad \psi'^2 = |1+w|S^2\rho_\psi.$$

For $w < -1$ we take the action to have a negative kinetic term, so

$$\psi'' + 2\mathcal{H}\psi' + D^a D_a\psi \pm S^2 V_{,\psi} = 0$$

etc, where $\pm$ is the sign of $1+w$. The derivatives needed are then

$$V_{,\psi} = \mp\frac{3(1-w)}{2}H\dot{\psi} \qquad S^2 V_{,\psi} = \mp\frac{3(1-w)}{2}\mathcal{H}\psi',$$

$$V_{,\psi\psi} = \mp\frac{3(1-w)}{2}\left[\dot{H} - \frac{3}{2}H^2(1+w)\right] \qquad S^2 V_{,\psi\psi} = \mp\frac{3(1-w)}{2}\left[\mathcal{H}' - \frac{1}{2}\mathcal{H}^2(5+3w)\right].$$

In many cases constant w is actually a very good approximation as far as the CMB is concerned, with [10]

$$w_{\text{eff}} \equiv \frac{\int da\Omega_\psi(a)w(a)}{\int da\Omega_\psi(a)}.$$

In quintessence domination $\rho \to \rho_\psi$ and $V_{,\psi\psi} \to \pm 9(1-w^2)H^2/2$.

Fluid equations

Following [11] the default CAMB module actually uses the fluid equations. For varying w these are

$$\delta_i' + 3\mathcal{H}(\hat{c}_{s,i}^2 - w_i)(\delta_i + 3\mathcal{H}(1+w_i)v_i/k) + (1+w_i)kv_i + 3\mathcal{H}w'v/k = -3(1+w_i)h' \tag{24}$$

$$v_i' + \mathcal{H}(1-3\hat{c}_{s,i}^2)v_i + kA = k\hat{c}_{s,i}^2\delta_i/(1+w_i)\,, \tag{25}$$

where hatted quantities are evaluated in the frame co-moving with the dark energy. These equations are implemented in the code with $w' = 0$, and are equivalent to using the above, with the additional possibility of using the rest-frame sound speed $\hat{c}$ different from one for generalized dark energy models ($\hat{c} = 1$ for quintessence)

V. TIGHT COUPLING

At early times the baryons and photons are tightly coupled; the opacity $\tau_c^{-1} \equiv S\mathrm{n}_e\sigma_\mathrm{T}$ is large. This means $\delta \equiv q_\gamma - 4/3v_b \approx 0$, and propagating the full photon hierarchy is impossible due to numerical problems in the source term proportional to $\tau_c^{-1}\delta$. The solution it to do an expansion in τ_c which is valid for $\epsilon \equiv \max(k\tau_c, \mathcal{H}\tau_c) \ll 1$. To first order in ϵ, assuming $c_s^2 \propto 1/S$ and dropping tiny terms in c_s^2,

$$v_b' = \frac{1}{1+R}\left(kc_s^2\Delta_b + \frac{k}{4}R(\Delta_\gamma - 2\beta_2\pi_\gamma) - \mathcal{H}v_b\right) \tag{26}$$

$$+\left(\frac{2\mathcal{H}}{1+R} + (\ln\tau_c^{-1})'\right)\frac{3R}{4(1+R)}\delta - \frac{R\tau_c}{4(1+R)^2}\left(4(\mathcal{H}'+\mathcal{H}^2)v_b + k\left[2\mathcal{H}\Delta_\gamma + \Delta_\gamma' - 4c_s^2\Delta_b'\right]\right) \tag{27}$$

$$\delta = \frac{\tau_c}{3(1+R)}\left(k\Delta_\gamma - 4kc_s^2\Delta_b + 4\mathcal{H}v_b\right) \tag{28}$$

$$\pi_\gamma = \frac{32}{45}k\tau_c(v_b+\sigma) \tag{29}$$

$$E_2 = \frac{\pi_\gamma}{4} \tag{30}$$

where $R \equiv 4\rho_\gamma/3\rho_b$. Note that here I use the sign for E_2 consistent with that in the code, which is related to the polarization tensor by a minus sign. To a good, *but not good enough*, approximation $\mathrm{n}_e \propto 1/S^3$ over the region of interest, so $(\ln\tau_c^{-1})' \approx -2\mathcal{H}$. However using this result does lead, for certain k, to a region in time in which neither the tight coupling approximation is accurate nor the full equations are stable. This is probably exacerbated by the RECFAST reionization history which doesn't change so abruptly at recombination. Using a numerical value for $(\ln\tau_c^{-1})'$ avoids the problem. Including the values for π_γ and E_2 makes the switch from tight coupling more robust, and allows it to be pushed to later times thereby speeding up the evolution on small scales.

VI. REGULAR INITIAL CONDITIONS

Following[1] [13] we define $\omega \equiv \Omega_m\mathcal{H}_0/\sqrt{\Omega_R}$, where $\Omega_R = \Omega_\gamma + \Omega_\nu$. The Friedmann equation gives

$$S = \frac{\Omega_m H_0^2}{\omega^2}\left(\omega\tau + \frac{1}{4}\omega^2\tau^2 - \frac{1}{6}K\omega\tau^3 - \frac{1}{48}K\omega^2\tau^4 + \mathcal{O}(\tau^5)\right).$$

Taking the lowest order in the tight coupling approximation we have the drag term

$$Sn_e\sigma_T(4/3v_b - q_\gamma) = -\frac{\rho_b}{3\rho_b + 4\rho_\gamma}(k\Delta_\gamma + 4\mathcal{H}v_b).$$

Higher orders corrections in $1/(n_e\sigma_T)$ are neglected (and hence π_γ and higher moments are zero, $v_b = 3q_\gamma/4$), and also assume $c_s^2 = p_b = 0$. We define $R_\nu = \Omega_\nu/\Omega_R$, $R_\gamma = \Omega_\gamma/\Omega_R$, $R_b = \Omega_b/\Omega_m$, $R_c = \Omega_c/\Omega_m$. The adiabatic mode (in

[1] The only difference here is that we give results for additional variables, and define the isocurvature modes following [12].

the CDM frame — remember χ is defined in terms of η in the comoving frame, $\bar{\eta} = \eta + \mathcal{O}(k^2\tau^2)$) is, for $\chi_0 = -1$,

$$\eta = 2\beta_2\left(1 - \frac{(k\tau)^2}{12}\left[\beta_2 - \frac{10}{4R_\nu + 15}\right]\right) \tag{31}$$

$$\Delta_\gamma = \Delta_\nu = \frac{\beta_2}{3}(k\tau)^2 - \frac{\beta_2}{15}\omega k^2\tau^3 \tag{32}$$

$$\Delta_c = \Delta_b = \frac{\beta_2}{4}(k\tau)^2 - \frac{\beta_2}{20}\omega k^2\tau^3 \tag{33}$$

$$q_\gamma = \frac{4}{3}v_b = \frac{\beta_2(k\tau)^3}{27} \tag{34}$$

$$q_\nu = \frac{\beta_2(k\tau)^3}{27}\frac{4R_\nu + 23}{4R_\nu + 15} \tag{35}$$

$$\pi_\nu = -\frac{4}{3}\frac{(k\tau)^2}{4R_\nu + 15} - \frac{\omega k^2\tau^3}{3}\frac{4R_\nu - 5}{(4R_\nu + 15)(2R_\nu + 15)} \tag{36}$$

$$G_3 = -\frac{4}{21}\frac{(k\tau)^3}{4R_\nu + 15} \tag{37}$$

Further quantities not needed in the code are

$$\chi = -1 + \frac{(k\tau)^2}{12}\left(\beta_2 + \frac{4R_\nu - 5}{4R_\nu + 15}\right) \tag{38}$$

$$\Psi = \frac{-10}{4R_\nu + 15} - \frac{25\omega\tau}{8}\frac{(8R_\nu - 3)}{(4R_\nu + 15)(2R_\nu + 15)} \tag{39}$$

$$\frac{\kappa S^2\Pi}{2k^2} = \frac{-2R_v}{4R_\nu + 15} + \frac{35}{2}\frac{\omega\tau R_\nu}{(2R_\nu + 15)(4R_\nu + 15)} \tag{40}$$

$$\sigma = \frac{-5k\tau}{4R_\nu + 15} - \frac{15\omega k\tau^2}{8}\frac{4R_\nu - 5}{(4R_\nu + 15)(2R_\nu + 15)} \tag{41}$$

$$\mathcal{Z} = \frac{3h'}{k} = -\frac{\beta_2}{2}k\tau + \frac{3\beta_2}{20}\omega k\tau^2. \tag{42}$$

Note that η and hence χ are constant to linear order, whereas the potential is only constant at zeroth order due to the matter changing the background equation of state.

CDM isocurvature mode:

$$\eta = \frac{\beta_2}{3}R_c\omega\tau - \frac{\beta_2}{8}R_c(\omega\tau)^2 \tag{43}$$

$$\Delta_c = 1 - \frac{1}{2}R_c\omega\tau + \frac{3}{16}R_c(\omega\tau)^2 \tag{44}$$

$$\Delta_b = -\frac{1}{2}R_c\omega\tau + \frac{3}{16}R_c(\omega\tau)^2 \tag{45}$$

$$\Delta_\gamma = \Delta_\nu = -\frac{2}{3}R_c\omega\tau + \frac{1}{4}R_c(\omega\tau)^2 \tag{46}$$

$$q_\gamma = q_\nu + \mathcal{O}(\tau^3) = -\frac{1}{9}R_c\omega k\tau^2 \tag{47}$$

$$\pi_\nu = -\frac{1}{3}\frac{R_c\omega k^2\tau^3}{2R_\nu + 15} \tag{48}$$

$$\Phi = \frac{1}{8}\frac{R_c(4R_\nu + 15)\omega\tau}{(2R_\nu + 15)} \tag{49}$$

$$\sigma = \frac{1}{24}\frac{R_c(4R_\nu - 15)\omega k\tau^2}{2R_\nu + 15} \tag{50}$$

There is a solution $R_c\Delta_c = -R_b\Delta_b$ with everything else zero, since in this case there is no density perturbation and hence the dynamics is that of the background. The baryon isocurvature mode is given by subtracting $(\Delta_c = 1, \Delta_b = -R_c/R_b)$ from the above mode, and is equivalent as far as the CMB is concerned so that $C_l(\text{CDM iso}, \Delta_{c0} = 1) = R_c^2/R_b^2 C_l(\text{baryon iso}, \Delta_{b0} = 1)$. The CDM isocurvature mode has a particularly simple form using the gauge invariant variables $\Delta_i^{\eta=0}$ (the perturbation in the frame in which $\eta = 0$, see later), with $\Delta_c^{\eta=0} = 1$, $\Delta_{\gamma,b,\nu}^{\eta=0} = 0$ to the above order.

We choose the neutrino isocurvature modes to have zero initial perturbation to the 3-Ricci scalar — a fairly natural definition of 'isocurvature'. This implies that the density perturbation in the comoving frame (proportional to Φ) is non-zero, which makes an alternative definition of isocurvature, in which the result is a sum of the mode below and the adiabatic mode. Neutrino isocurvature density mode:

$$\Delta_\gamma = -\frac{R_\nu}{R_\gamma} + \frac{R_\nu}{6R_\gamma}(k\tau)^2 \tag{51}$$

$$\Delta_\nu = 1 - \frac{(k\tau)^2}{6} \tag{52}$$

$$\Delta_c = -\frac{\omega k^2\tau^3}{80}\frac{R_\nu R_b}{R_\gamma} \tag{53}$$

$$\Delta_b = \frac{1}{8}\frac{R_\nu (k\tau)^2}{R_\gamma} \tag{54}$$

$$q_\gamma = -\frac{R_\nu}{3R_\gamma}k\tau + \frac{\omega k\tau^2}{4}\frac{R_\nu R_b}{R_\gamma^2} \tag{55}$$

$$q_\nu = \frac{k\tau}{3} - \frac{(k\tau)^3}{54}\left(1 + \frac{12\beta_2}{4R_\nu + 15}\right) \tag{56}$$

$$\pi_\nu = \frac{(k\tau)^2}{4R_\nu + 15} \tag{57}$$

$$\eta = \frac{\beta_2 (k\tau)^2}{3}\frac{R_\nu}{4R_\nu + 15} \tag{58}$$

$$\Phi = -\frac{R_\nu}{4R_\nu + 15} - \frac{\omega\tau}{4}\frac{R_\nu(2R_\nu - 15)}{(4R_\nu + 15)(2R_\nu + 15)} \tag{59}$$

The weird neutrino velocity isocurvature mode is

$$\Delta_\gamma = k\tau\frac{R_\nu}{R_\gamma} - \frac{3\omega k\tau^2}{16}\frac{R_b(2 + R_\gamma)}{R_\gamma^2} \tag{60}$$

$$\Delta_\nu = -k\tau - \frac{3\omega k\tau^2 R_b}{16R_\gamma} \tag{61}$$

$$\Delta_c = -\frac{9\omega k\tau^2}{64}\frac{R_\nu R_b}{R_\gamma} \tag{62}$$

$$\Delta_b = \frac{3R_\nu}{4R_\gamma}k\tau - \frac{9\omega k\tau^2}{64}\frac{R_b(2 + R_\gamma)}{R_\gamma^2} \tag{63}$$

$$q_\gamma = -\frac{R_\nu}{R_\gamma} + \frac{3R_\nu R_b}{4R_\gamma^2}\omega\tau + \frac{(k\tau)^2}{6}\frac{R_\nu}{R_\gamma} + \frac{3(\omega\tau)^2}{16}\frac{R_\nu R_b}{R_\gamma^3}(R_\gamma - 3R_b) \tag{64}$$

$$q_\nu = 1 - \frac{(k\tau)^2}{6}\left(1 + \frac{4\beta_2}{4R_\nu + 5}\right) \tag{65}$$

$$\pi_\nu = \frac{2k\tau}{(4R_\nu + 5)} + \omega k\tau^2\frac{6}{(4R_\nu + 5)(4R_\nu + 15)} \tag{66}$$

$$G_3 = \frac{3}{7}\frac{(k\tau)^2}{4R_\nu + 5} \tag{67}$$

$$\eta = 2\beta_2 k\tau\frac{R_\nu}{4R_\nu + 5} + \omega k\tau^2\frac{3\beta_2 R_\nu}{32}\left(\frac{R_b}{R_\gamma} - \frac{80}{(4R_\nu + 5)(4R_\nu + 15)}\right) \tag{68}$$

$$\Phi = \Psi + \mathcal{O}(1) = -\frac{3R_\nu}{4R_\nu + 5}(k\tau)^{-1} \tag{69}$$

$$\phi = \frac{45}{4}\frac{R_\nu\omega}{k(4R_\nu + 15)(4R_\nu + 5)} \tag{70}$$

$$\sigma = -3\frac{R_\nu}{4R_\nu + 5} + 2\phi k\tau \tag{71}$$

Note that the variables Φ and Ψ are singular. However these are only natural variables in the zero shear frame (the curvature and acceleration respectively), whereas ϕ is frame invariant and is regular. The results above generalize those of [12] for non-flat models (note that the adiabatic anisotropic stress term in [12] is wrong, as is the neutrino octopole in the neutrino velocity mode).

The quintessence field ψ is strongly damped by the high expansion rate in the early universe, and thus rapidly obtains very small velocity. Choosing the origin of the field $\psi(0) = 0$ we have $\psi(\tau) = -H_0^2\Omega_R V_{,\psi}\tau^4/20 + \mathcal{O}(\tau^5)$, $w_\psi = -1 + H_0^2\Omega_R V_{,\psi}^2\tau^4/(25V)$, and the above modes are unchanged on adding a quintessence field. There is an additional quintessence isocurvature mode with

$$\mathcal{V} = \frac{V}{V_{,\psi}}\left(1 - \frac{(k\tau)^2}{6} + \frac{\omega k^2\tau^3}{72}\right) \tag{72}$$

$$\Delta_\psi = 1 - \frac{(k\tau)^2}{10} \tag{73}$$

and the other perturbations zero to this order. Note that fixing $w = \text{const}$ is inconsistent with these assumptions. However in the early universe the evolution is independent of the potential, so using the above initial conditions with $w = \text{const}$ is potentially not a bad approximation for various classes of quintessence models which start with $w \sim -1$, then evolve to have an effective constant $w = w_{\text{eff}}$.

The regular vector mode solution is given in Ref. [14] and the mode with magnetic fields in Ref. [8].

Frame invariant series

The isocurvature modes take a particularly simple form in the frame of identically vanishing curvature, in other words using the frame invariant variables $\hat{\Delta}_i = \Delta_i + \frac{3}{2\beta_2}(1 + w_i)\eta$, e.g. CDM isocurvature:

$$\hat{\Delta}_c = 1 - \frac{1}{72}\frac{R_c(4R_\nu - 15)\omega k^2\tau^3}{2R_\nu + 15} \tag{74}$$

$$\hat{\Delta}_\gamma = \hat{\Delta}_\nu = \frac{5}{6}\frac{R_c\omega k^2\tau^3}{2(R_\nu + 15)} \tag{75}$$

$$\hat{\Delta}_b = \frac{5}{8}\frac{R_c\omega k^2\tau^3}{2(R_\nu + 15)} \tag{76}$$

$$v_b + \sigma = -\frac{15}{8}\frac{R_c\omega k\tau^2}{2R_\nu + 15} \tag{77}$$

$$v_c + \sigma = \frac{1}{24}\frac{R_c(4R_\nu - 15)\omega k\tau^2}{2R_\nu + 15} \tag{78}$$

$$q_\nu + \frac{4}{3}\sigma = -\frac{5}{2}\frac{R_c\omega k\tau^2}{2R_\nu + 15} \tag{79}$$

$$\Psi = \frac{1}{8}\frac{R_c(4R_\nu - 15)\omega\tau}{2R_\nu + 15} \tag{80}$$

(equal signs at this order only), consistent with comments below about conservation of the density perturbations in the zero curvature frame. See also Ref. [15].

VII. TENSORS

The power spectrum defined by the transverse traceless part of the metric tensor so that

$$\langle h_{ab}h^{ab}\rangle = \sum_\nu \frac{\nu}{\nu^2 - 1}\frac{\nu^2 - 4}{\nu^2}\mathcal{P}_h(\nu)$$

where the $\nu^2 - 4$ factor is the mode sum $l = 2\ldots\nu - 1$ of $2l + 1$. Note that $\nu^2/(\nu^2 - 1)$ is no longer the measure over $\ln k$ for tensor modes. The CMB result is then

$$C_l = \frac{\pi}{32}\frac{(l+1)(l+2)}{2l(l-1)}\sum_\nu \frac{\nu}{\nu^2 - 1}\frac{\nu^2 - 3}{\nu^2}\mathcal{P}_h(\nu)[T_I^{(l)}(\nu)]^2.$$

Internally CAMB uses the H_k, the metric tensor variable, and the shear σ_k. The relation between H_k, E_k (the electric part of the Weyl tensor) and the shear σ_k is

$$H_k = -\frac{\nu^2 - 3}{\nu^2 - 1}(2E_k + \sigma_k'/k)$$

and $H_k' = -k\sigma_k$. Using the series solution the relation between the initial E_k and H_k is

$$E_k = -\frac{2R_\nu + 10}{4R_\nu + 15}\frac{\nu^2 - 1}{\nu^2 - 3}H_k$$

where R_ν is the ratio of the neutrino and total radiation densities.

VIII. MATTER POWER SPECTRUM

σ_8 is defined by

$$\sigma_8^2 = \frac{\langle|\int dV_3 \Delta(x)|^2\rangle}{|\int dV_3|^2}$$

where Δ is the fractional total matter density perturbation (frame irrelevant at late times), and the integral is over a $8h^{-1}$Mpc sphere. In spherical polar harmonics we have

$$\Delta = \sum_{lm} \int d\ln k \Delta_{klm} Y_{lm} j_l(kr),$$

then, since $Y_{00} = 1/\sqrt{4\pi}$,

$$\begin{aligned}\frac{\int dV_3 \Delta(x)}{\int dV_3} &= \frac{3}{4\pi r^3}\int d\ln k \int_0^r dr 4\pi r^2 \Delta_{k00}\frac{j_0(kr)}{\sqrt{4\pi}} \\ &= \frac{3}{\sqrt{4\pi}}\frac{kr\cos kr - \sin kr}{k^3 r^3}\Delta_{k00}\end{aligned}$$

where $r = 8h^{-1}$Mpc. Now by assumed statistical isotropy

$$\langle\Delta^2(x)\rangle = \langle\Delta^2(0)\rangle = \frac{1}{4\pi}\int d\ln k \int d\ln k' \langle\Delta_{k00}\Delta_{k'00}\rangle = \int d\ln k \mathcal{P}_\Delta$$

It follows that

$$\langle\Delta_{klm}\Delta_{k'l'm'}\rangle = 4\pi k \delta_{ll'}\delta_{mm'}\delta(k - k')\mathcal{P}_\Delta(k)$$

and hence

$$\sigma_8^2 = \int d\ln k \left[3\frac{kr\cos kr - \sin kr}{k^3 r^3}\right]^2 T_\Delta^2 \mathcal{P}_\chi$$

where T_Δ is the transfer function.

We define the matter power spectrum so that

$$\langle\Delta^2(x)\rangle = \frac{1}{(2\pi)^3}\int d^3k P_k = \int d\ln k \left(\frac{k^3 P_k}{2\pi^2}\right)$$

and is usually expressed in units of h^{-1}Mpc. Therefore

$$P_k = \frac{2\pi^2 \mathcal{P}_\chi}{k^3}T_\Delta^2.$$

For COBE-normalized runs this can be compared with the output from CMBFAST using

$$P_k = 8\pi^3 h^3 k (TF)^2 \mathrm{d2norm}$$

where TF is the quantity in the transfer function output file, and k is just k (the first column of the output file is k/h).

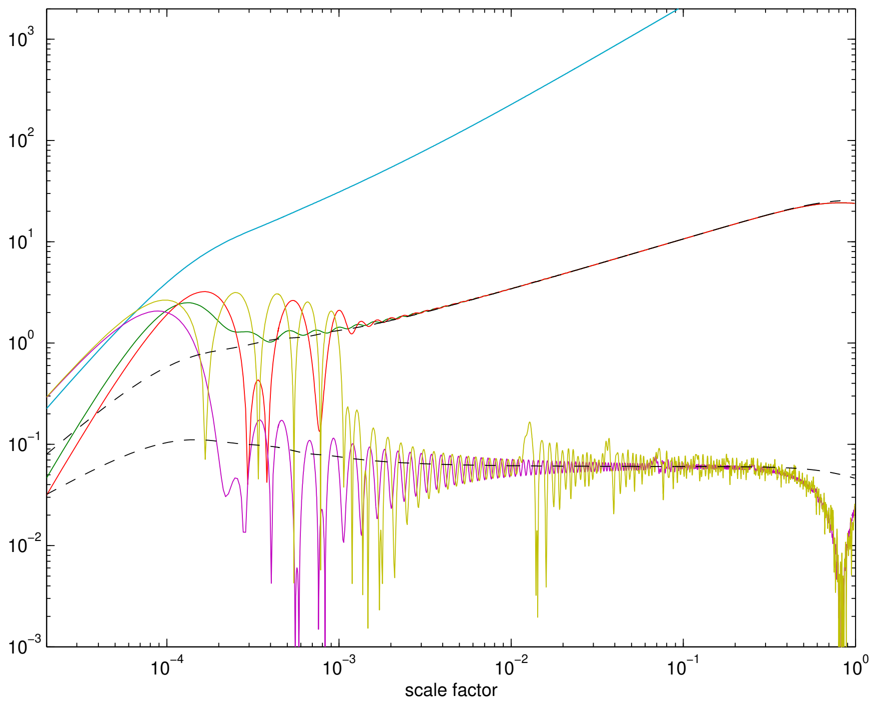

FIG. 1: Absolute value $k = 0.1\text{Mpc}^{-1}$ mode evolution of Δ_c (top), Δ_γ and Δ_ν (bottom), q_γ and q_ν (middle), together with the dotted lines which are the approximations used after matter domination.

IX. THE CAMB CODE

The physics is contained in the equations.f90 file. Everything is in conformal time and units of megaparsecs. The dtauda routine returns $1/a'$ (scale factor $a = S$), where the conformal Hubble rate is a'/a. This is used to compute the conformal age of the universe, and background evolution for the pre-computed ionization history and should always be finite. Background density variables include a factor of a^2, like grho $\equiv \kappa a^2 \rho$.

The initial conditions for the scalars and tensors are in the 'initial' and 'initialt' routines (see above for details). The derivs and derivst functions return the differential equations - a vector y' of derivatives of the variables being evolved. The fderivs and fderivst routines are very similar and used for flat models.

The 'output' and 'outputt' routines calculate the sources for integration against the Bessel functions.

X. LATE TIME EVOLUTION

To compute the matter transfer functions at late time on small scales many oscillations of the photon and neutrino multipoles have to be integrated. The integration is in fact very inaccurate except on high accuracy settings because of the low-l truncation of the multipole equations. The effect after matter domination is rather small so this is a waste of time. However even though the oscillations are inaccurate, the mean values are non-zero (and about correct), so for accurate results it is important to capture the non-oscillatory behaviour of the velocities when the radiation density fraction is not entirely negligible.

Ref. [16] suggest one scheme for cutting off this evolution. In CAMB we are using the synchronous gauge, and instead use the following analytic solutions for the non-oscillatory zero-acceleration frame evolution

$$\Delta_\gamma \approx \Delta_\nu \approx \frac{2\kappa S^2}{3k^2}(\rho_c \Delta_c + \rho_b \Delta_b) \tag{81}$$

$$q_\gamma \approx q_\nu \approx \frac{2k}{3} \frac{\Delta_\gamma}{\sqrt{\kappa S^2[\rho_b + \rho_c]/3}} \tag{82}$$

ones the modes are well inside the horizon, after recombination, and well in to matter domination. The higher moments are set to zero. The particular representation of q_γ here has been chosen so that it works quite well through the matter-radiation transition. On scales with $k \lesssim 0.02\text{Mpc}^{-1}$ we can't use this truncation because of reionization, but on smaller scales it can save a significant amount of time, especially when computing the matter power spectrum on small scales. Parameters are chosen to keep the matter power spectrum accurate to $< 0.3\%$ on standard accuracy settings. See Fig. 1. The approximation to Δ_γ is fairly irrelevant, but the velocities are crucial to obtain sub-percent level precision with a switch to the approximate solution at $a > 10a_{\text{eq}}$ for modes well inside the horizon. We switch off the higher multipoles suddenly rather than using a smooth transition as in Ref. [16], so the multipole equations do not need to be propagated at all after the transition.

Note the the synchronous (zero acceleration frame) *photon* density perturbations are not the same as the Newtonian one, $\Delta_\gamma - 4\mathcal{H}\sigma/k$, amounting to a factor of three difference during matter domination. The photon velocities are also quite different, with the Newtonian gauge $q_\gamma + 4/3\sigma \sim 0$.

XI. MASSIVE NEUTRINOS, LENSING ETC

These are properly documented! The treatment of massive neutrinos is discussed in Ref. [7]. The CMB lensing method is described in Ref. [17].

XII. DIFFERENT VARIABLES

The following is not strictly related to CAMB, but useful maybe to understand the equations.

As in my thesis I use total matter variables, e.g. $q = \sum_i \rho_i q_i$, $\mathcal{X} = \sum_i \rho_i \Delta_i$. In the co-moving $q = 0$ (zero heat flux, denoted by a bar) frame

$$\bar{\mathcal{X}} = \mathcal{X} + 3\mathcal{H}q/k \qquad \bar{\sigma} = \sigma + \frac{q}{\rho + p} \qquad \beta_2\bar{\sigma} = \bar{\mathcal{Z}}$$

are frame invariant, and hence

$$\bar{\eta} = \eta - \frac{2\beta_2\mathcal{H}q}{k(\rho + p)}.$$

Using the frame invariance of $\bar{\mathcal{X}}$

$$\hat{q} = q + k\mathcal{X}/3\mathcal{H}$$

where a hat denotes the $\mathcal{X} = 0$ frame, and hence

$$\hat{\eta} = \eta + \frac{2\beta_2\mathcal{X}}{3(\rho + p)} = \mathcal{H}\left(\frac{\eta}{\mathcal{H}} - \frac{2\beta_2\mathcal{X}}{\dot{\rho}}\right)$$

is also frame invariant, so $\hat{\eta} = \eta_{\mathcal{X}=0} = \frac{2\beta_2}{3(\rho+p)}\mathcal{X}_{\eta=0}$. In terms of the flat metric variables as used by [18]

$$-\frac{1}{2}\hat{\eta} \sim -\zeta = \mathcal{H}\xi \equiv \mathcal{H}\left(\frac{\psi}{\mathcal{H}} + \frac{\delta\rho}{\dot{\rho}}\right)$$

and $\eta \sim -2\psi$. Since $\mathcal{X}/\dot{\rho}$ transforms like $\mathcal{X}_i/\dot{\rho}_i$ the curvature perturbation in the frame $\mathcal{X}_i = 0$ is the frame invariant quantity

$$\eta - 2\beta_2\mathcal{H}\frac{\mathcal{X}_i}{\dot{\rho}_i}.$$

The energy conservation equation gives

$$3\hat{h}'_a(\rho + p) = -3\mathcal{H}\hat{\mathcal{X}}^p_a - S^2\hat{D}_a\hat{D}^b\hat{q}_b.$$

which implies that $\hat{h}_a$ is constant on large scales where the derivatives can be neglected if the pressure perturbation in the uniform energy density frame is zero, i.e. $\hat{\mathcal{X}}^p_a = (D_a p)_{\text{nad}} \equiv \mathcal{X}^p_a - p'/\rho'\mathcal{X}_a = 0$. As emphasised by [18] this follows purely from energy conservation, independently of the field equation (though in different theories the energy that is conserved may include components from the different theory). Using the general result

$$\eta' = -\frac{2\beta_2}{3}(k\sigma - 3h')$$

we have

$$\frac{1}{2\beta_2}\hat{\eta}' = -\mathcal{H}\frac{\delta p_{\rm nad}}{\rho + p} - \frac{k\bar{\sigma}}{3}.$$

Hence on large scales $\hat{\eta}$ is conserved if $K = 0$ and $\delta p_{\rm nad} = 0$. This applies equally well if the total densisties are replaced with those for an individual species as long as they do not interact. For adiabatic modes $\hat{\eta} = \bar{\eta} + \bar{\mathcal{X}}\frac{2\beta_2}{3(\rho+p)} \approx \bar{\eta}$ on large scales since $\bar{\mathcal{X}}$ is suppressed by a factor of k^2. The total and individual results are related by

$$\hat{\eta} = \frac{\sum_i(\rho_i + p_i)\hat{\eta}_i}{\sum_i(\rho_i + p_i)}.$$

Note that from the η' equation above, in the gauge in which $h' = 0$, η is conserved on large scales no matter what. Is this a usefull statement, or like saying that in the $h' = 0$ gauge h is conserved !? Wands etc define this gauge by as the zero number density perturbation gauge, which amounts to the same thing as if $h' = 0$ the local volume element does not change with time and can be chosen to be zero initially. See also Ref. [15].

Newtonian Gauge

The Newtonian gauge is the frame with zero shear: $\sigma = 0$. To construct Newtonian gauge quantities from those in any other frame you just need to identify the gauge-invariant combinations that give what you want when the shear vanishes. See the Change of Frame section later for help constructing frame invariant combinations. In particular the Newtonian gauge velocity and density perturbations are given by

$$v_N^{(i)} = v^{(i)} + \sigma_i \qquad \Delta_N^{(i)} = \Delta^{(i)} - 3\mathcal{H}(1 + w^{(i)})\sigma/k$$

where $w = p/\rho$. e.g. $\Delta_N^\gamma = \Delta^\gamma - 4\mathcal{H}\sigma/k$. In general

$$\Phi = \frac{\eta}{2\beta_2} + \frac{\mathcal{H}\sigma}{k}$$

and hence $2\beta_2\Phi = \eta_N$ is the curvature in the zero shear frame (Newtonian gauge). From the σ' equation $A_N = -\Psi$, and the η' relation $\eta_N' = 2\beta_2 h_N'$. A useful relation is

$$\left(\Delta_i + \frac{3(1 + w_i)}{2\beta_2}\eta\right)' = -k(1 + w_i)(v_i + \sigma).$$

See also the appendix of Ref. [15] and `http://cosmocoffee.info/viewtopic.php?t=212`.

XIII. NEUTRINO DAMPING ON TENSORS

It's well known that the effect of neutrino anisotropic stress can be neglected when computing the large scale temperature anisotropy. This is because the dominant large scale contribution ($\ell < 150$) comes from the anisotropy generated by viewing an isotropic last scattering surface through gravitational waves along the line of sight. The gravitational waves source photon anisotropies via their derivative $H' \sim \sigma$. On super-horizon scales $H' = 0$ and there is no contribution. On sub-horizon scales the waves decay, therefore the main contribution to the large scale anisotropy comes from when the waves come inside the horizon and start to decay, but before they start oscillating at which point they become small. After last scattering the evolution is matter dominated, so the neutrinos have almost no contribution to the energy density and hence the effect of neutrino anisotropic stress is negligible. For this reason CMBFAST, and by default CAMB, neglect the neutrinos as they slow down the computation.

For polarization things are different, as in this case the linear contribution to the anisotropy comes predominantly from the photon anisotropic stress sourced by H'. The large scale signal due to reionization is however insensitive

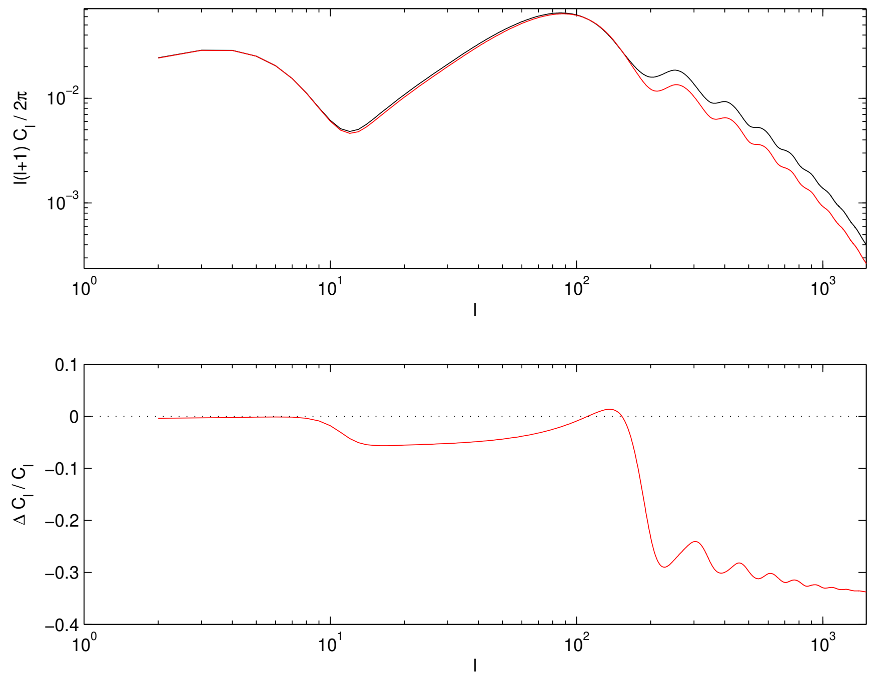

FIG. 2: The BB power spectrum with and without including the effect of neutrino damping (CAMB's do_tensor_neutrinos parameter)

to neutrinos as this is during matter domination, thus the polarization signal with highest signal to noise is also unaffected. On smaller scales the main contribution comes from the photon anisotropic stress at last scattering, which is affected by H', itself a function of the damping due to neutrinos.

Ref. [19] has analysed the effect of neutrinos semi-analytically. In particular he finds a general semi-analytical result for modes which enter the horizon deep in radiation domination. Since this is before last scattering, and once inside the horizon the modes decay, this regime corresponds to the small oscillatory signal at high $\ell \gtrsim 200$. This is of little consequence for observations, though the effect is a substantial 35% reduction in power.

The polarization power spectra peak at $\ell \sim 100$, corresponding to modes which have maximal velocity at the time of last scattering. Therefore the most interesting region (apart from reionization) is on intermediate scales. Ref. [19] also analyses the case of modes which enter the horizon after radiation domination, and claims a $\sim 10\%$ damping of the power by analysing the effect of neutrinos on H' at last scattering. This accounts for the damping of individual modes and the phase difference relative to the fixed time of recombination.

See Figure. 2 for CAMB's output. The large scale effect of $\sim 5\%$ on $\ell < 100$ is numerically smaller than that in Ref. [19], and for $l \gtrsim 100$ the neutrinos actually increase the power slightly. My understanding is this: A mode which (without neutrinos) reaches maximal velocity at some time shortly after last scattering may be slowed by the damping such that the velocity *at the time of last scattering* is *higher* than without the damping, even though the overall amplitude of the mode is lower. Thus for modes on larger scales than the peak $l <\sim 100$ there should be a damping effect, but for $l \gtrsim 100$ there can be an increase in power at that ℓ

(CAMB has always supported tensor neutrinos by a flag in equationsxxx.f90; as of Dec 2003 this can also be set from the parameter input file)

XIV. CHANGE OF FRAME

In the frame $\tilde{u} = u + v$ we have

$$
\begin{aligned}
\tilde{q} &= q - (\rho + p)v \\
\tilde{\mathcal{X}} &= \mathcal{X} - \frac{\rho' v}{k} \\
\tilde{\eta} &= \eta - \frac{2\beta_2 \mathcal{H} v}{k} \\
\tilde{\sigma} &= \sigma + v \\
\tilde{A} &= A + \frac{v' + \mathcal{H}v}{k} \\
\tilde{h}' &= h' + \frac{kv}{3} - \frac{(\mathcal{H}v)'}{k}.
\end{aligned}
$$

In terms of the covariant tensors we have (not checked for non-flat)

$$
\begin{aligned}
\tilde{\eta}_a &= \eta_a - 2\mathcal{H} D_a D^b v_b \\
\tilde{h}'_a &= h'_a + \frac{1}{3} S^2 D_a D^b v_b - (\mathcal{H} v_a)' \\
\tilde{\delta\rho}_a &= \delta\rho_a - \rho' v_a \\
\tilde{\sigma}_{ab} &= \sigma_{ab} + D_{\langle a} v_{b\rangle}
\end{aligned}
$$

(note notation mess: $\delta\rho \equiv \mathcal{X}$) so e.g. a frame invariant curvature perturbation is

$$
\eta_a - 2\mathcal{H} \frac{D_a D^b \delta\rho_b}{\rho'}. \tag{83}
$$

XV. ACTION (E.G. CLOSED INSTANTON)

We perform a harmonic expansion of a quantity X in the form

$$
X = \sum_{klm} X_{klm} Q_{klm}
$$

where $\Delta Q_{klm} = -k^2 Q^k$ and we choose $Q_{klm} = \Phi_l^\nu(\chi) Y_{lm}$ where we normalize the hyper-spherical Bessel functions so that $\Phi_0^\nu(0) = 1$ and $k^2 = K(\nu^2 - 1)$ (for closed models: everywhere $\nu^2 - 1 \to \nu^2 + 1$ for open models). For flat models $\Phi_l^\nu(r) = j_l(kr)$.

We expand a variable X in the form

$$
X = \sum_k X_k Q^k
$$

where k labels l,m and ν and $\Delta Q^k = -k^2 Q^k$. Mode expansion in the action gives

$$
S = \frac{1}{2} \int dV_4 X \mathcal{L} X = \frac{N}{2} \int d\tau \sum_k X_k \mathcal{L} X_k
$$

where

$$
\int dV_3 Q^k Q^{k'} = N \delta_{kk'}
$$

where $\delta_{kk'}$ is defined by $\sum_k \delta_{kk'} X_k = X_{k'}$. Spatial derivatives in $\mathcal{L}$ are converted to ks. In a compact space the sum over modes is of the form

$$
\sum_k = \sum_{\nu l m} A_\nu
$$

where A_ν is some arbitrary ν-dependent factor. The expectation value of the mode coefficients is therefore given by

$$\langle |X_k(\tau)|^2\rangle = \frac{1}{NA_\nu}G(\tau,\tau)$$

where the Greens' function is unambiguously determined by the real space operator and satisfies

$$\mathcal{L}G(\tau,\tau') = \delta(\tau-\tau').$$

The expectation value of each ν mode will be the same and

$$\langle |X(x,\tau)|^2\rangle = \frac{\mathcal{K}^{3/2}}{2\pi^2}\sum_k G(\tau,\tau)/A_\nu = \sum_\nu \frac{\nu}{\nu^2-1}\mathcal{P}(\nu)$$

where

$$\mathcal{P}(\nu) \equiv \frac{\mathcal{K}^{3/2}\nu(\nu^2-1)}{2\pi^2}G(\tau,\tau)$$

is the power spectrum. The CMB power spectrum is given by

$$C_l = \frac{1}{16}\frac{1}{2l+1}\sum_k \frac{2N\mathcal{K}^{3/2}\nu^2}{\pi}\langle |X_k(\tau_r)|^2\rangle [T_I^{(l)}(\nu)]^2$$

where $T_I^{(l)}(\nu)$ is the transfer function giving I_l when $X_k(\tau_r)=1$ and r denotes the early radiation dominated era when the mode is still well outside the horizon. This gives

$$C_l = \frac{1}{16}\frac{2}{\pi}\sum_\nu \mathcal{K}^{3/2}\nu^2 G(\tau_r,\tau_r)[T_I^{(l)}(\nu)]^2 = \frac{4\pi}{16}\sum_\nu \frac{\nu}{\nu^2-1}\mathcal{P}_r(\nu)[T_I^{(l)}(\nu)]^2.$$

For the CMB Anthony uses $A_\nu = \mathcal{K}^{3/2}\nu^2$ and $N=\pi/2$ where $Q^k = \Phi_l^\nu(\chi)Y_{lm}$ and $\Phi_0^\nu(0)=1$.

In the tensor case we have an action of the form

$$S = \frac{1}{8\kappa}\int dV_4 h^{ab}\mathcal{L}h_{ab}$$

and write

$$h_{ab} = \sum_k 2H_k Q^k_{ab}$$

where k labels ν,l,m and P, the parity. The power spectrum is given by

$$\mathcal{P}_h(\nu) \equiv \frac{4}{\pi^2}\mathcal{K}^{3/2}\nu(\nu^2-1)G(\tau,\tau)$$

where $\mathcal{L}G(\tau,\tau') = \delta(\tau-\tau')$. Note the additional factor of two difference from the additional sum over the two parities.

The contribution to the CMB is

$$C_l = \frac{1}{16}\frac{(l+1)(l+2)}{2l(l-1)}\frac{1}{2l+1}\sum_k \frac{2N\mathcal{K}^{3/2}\nu^2}{\pi}\frac{(\nu^2-3)}{\nu^2}\langle H_k^2\rangle [T_I^{(l)}(\nu)]^2$$

where k labels positive parity modes only ($\bar{Q}_{A_l}|_0 = 0$) giving the previous result.

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