Chapter 7

Angular Momentum, Part I (Constant $\hat{\mathrm{L}}$)

The angular momentum of a point mass, relative to a given origin, is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{7.1}$$

For a collection of particles, the total $\mathbf{L}$ is simply the sum of the $\mathbf{L}$'s of each particle (using the same origin for all the particles). We choose to look at this quantity $\mathbf{r} \times \mathbf{p}$ because we can make many useful statements about it. Some such statements were given at the beginning of Chapter 6. Later in this chapter we'll introduce the *torque*, $\boldsymbol{\tau}$, which appears in the bread-and-butter statement, $\boldsymbol{\tau} = d\mathbf{L}/dt$ (which is analogous to Newton's $\mathbf{F} = d\mathbf{p}/dt$ law). This equation is the basic ingredient, along with $\mathbf{F} = m\mathbf{a}$, in solving any rotation problem.

There are two basic types of angular momentum problems in the world. The solution to any rotational problem invariably comes down to using $\boldsymbol{\tau} = d\mathbf{L}/dt$, so we must determine how $\mathbf{L}$ changes in time. Since $\mathbf{L}$ is a vector, it can change because (1) its length changes, or (2) its direction changes (or through some combination of these effects). In other words, if we write $\mathbf{L} = L\hat{\mathbf{L}}$, where $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction, then $\mathbf{L}$ can change because L changes, or because $\hat{\mathbf{L}}$ changes, or both.

The first of these cases, that of constant $\hat{\mathbf{L}}$, is the easily understood one. If you have a spinning record (in which case $\mathbf{L} = \sum \mathbf{r} \times \mathbf{p}$ is perpendicular to the record), and if you give the record a tangential force in the proper direction, then it will speed up (in a precise way which we will soon determine). There is nothing mysterious going on here. If you push on the record, it goes faster. $\mathbf{L}$ points in the same direction as before, but simply has a larger magnitude. In fact, in this type of problem, you can completely forget that $\mathbf{L}$ is a vector; you can just deal with its magnitude L, and everything will be fine. This first case will be the subject of the present chapter.

In contrast, the second case, where $\mathbf{L}$ changes direction, can get rather confusing. This will be the subject of the following chapter, where we will talk about gyroscopes, tops, and other such spinning things that have a tendency to make one's head spin also. In this case, the entire point is that $\mathbf{L}$ is actually a vector. And unlike in the first case, you really have to visualize things in three dimensions to see what's going on.[1]

The angular momentum of a point mass is easy to calculate. The difficulty arises when we try to calculate the angular momentum of an extended body.

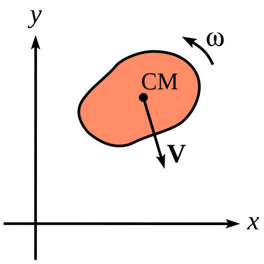

Figure 7.1

7.1 Pancake object in x-y plane

Consider a flat, rigid body undergoing arbitrary motion in the x-y plane (see Fig. 7.1). What is the angular momentum of this body, relative to the origin of the coordinate system?[2]

If we imagine the body to consist of particles of mass m_i, then the angular momentum of the entire body is the sum of the angular momenta of each m_i (which are $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$). So the total angular momentum is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i. \tag{7.2}$$

(For a continuous distribution of mass, we'd have an integral instead of a sum.) This $\mathbf{L}$ depends on the locations of the masses, and on the motion of the body, that is, how fast it is translating and spinning. Our goal is to find this dependence. This result will involve the geometry of the body in a specific way, as we will show.

In this chapter, we will deal only with pancake-like objects which move in the x-y plane (or simple extensions of these). We will find $\mathbf{L}$ relative to the origin, and we will also derive an expression for the kinetic energy.

Note that since both $\mathbf{r}$ and $\mathbf{p}$ for our pancake-like objects always lie in the x-y plane, the vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ always points in the $\hat{\mathbf{z}}$ direction. This fact is what makes this case easy to deal with; $\mathbf{L}$ changes only because its length changes, not its direction. So when we eventually get to the $\boldsymbol{\tau} = d\mathbf{L}/dt$ equation, it will take on a simple form.

Let's first look at a special case, and then we will look at general motion in the x-y plane.

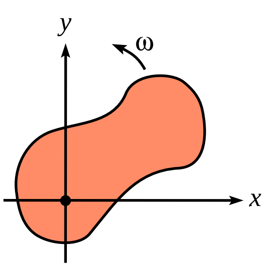

Figure 7.2

7.1.1 Rotation about the z-axis

The pancake in Fig. 7.2 rotates with angular speed ω around the z-axis, in the counterclockwise direction (as viewed from above). Consider a little piece of the body, with mass dm and position (x, y). Let $r = \sqrt{x^2 + y^2}$. This little piece travels in a circle around the origin. Its speed is[3] $v = \omega r$. Therefore, the angular momentum

[1] The difference between these two cases is essentially the same as the difference between the two basic $\mathbf{F} = d\mathbf{p}/dt$ cases. The vector $\mathbf{p}$ can change simply because its magnitude changes, in which case we have $F = ma$. Or, $\mathbf{p}$ can change because its direction changes, in which case we end up with the centripetal acceleration statement, $F = mv^2/r$. (Or, there could be a combination of these effects). The former case seems a bit more intuitive than the latter.

[2] Remember, $\mathbf{L}$ is defined relative to a chosen origin (since it has the vector $\mathbf{r}$ in it), so it makes no sense to ask what $\mathbf{L}$ is, without specifying what origin you've chosen.

[3] The velocity is actually given by $\mathbf{v} = \vec{\omega} \times \mathbf{r}$, which reduces to $v = \omega r$ in our case. The vector $\vec{\omega}$ is the *angular velocity vector*, which is defined to point along the axis of rotation, with magnitude ω

of this piece (relative to the origin) is equal to $\mathbf{L} = \mathbf{r} \times \mathbf{p} = r(v\,dm)\hat{\mathbf{z}} = dm\,r^2\omega\hat{\mathbf{z}}$. The $\hat{\mathbf{z}}$ direction arises from the cross product of the (orthogonal) vectors $\mathbf{r}$ and $\mathbf{p}$. The angular momentum of the entire body is therefore

$$\begin{aligned} \mathbf{L} &= \int r^2\omega\hat{\mathbf{z}}\,dm \\ &= \int (x^2+y^2)\omega\hat{\mathbf{z}}\,dm, \end{aligned} \tag{7.3}$$

where the integration runs over the area of the body. (If the density of the object is constant, as is usually the case, then we have $dm = \rho\,dx\,dy$.) If we define the *moment of inertia* around the z-axis as

$$I_z \equiv \int r^2\,dm = \int (x^2+y^2)\,dm, \tag{7.4}$$

then the z-component of $\mathbf{L}$ is

$$L_z = I_z\omega, \tag{7.5}$$

and L_x and L_y both equal zero.

In the case where the rigid body is made up of a collection of point masses, m_i, in the x-y plane, eq. (7.4) is simply

$$I_z \equiv \sum_i m_i r_i^2. \tag{7.6}$$

Given any rigid body in the x-y plane, we can calculate I_z. And given ω, we may then calculate L_z. In Section 7.2.1, we will calculate many moments of inertia, for practice.

What is the kinetic energy of our object? We need to add up the energies of all the little pieces. A little piece has energy $dm\,v^2/2 = dm(r\omega)^2/2$. So the total kinetic energy is

$$T = \int \frac{1}{2}r^2\omega^2\,dm. \tag{7.7}$$

With our definition of I_z in eq. (7.4), we have

$$T = \frac{1}{2}I_z\omega^2. \tag{7.8}$$

This is easy to remember, because it looks a lot like the kinetic energy of a point mass, $(1/2)mv^2$.

REMARK: Eqs. (7.4) and (7.5) actually give the correct result for L_z for any object, as long as the axis of rotation points in the z-direction. We don't need the pancake restriction. If the object has height in the z-direction, then the integral in eq. (7.4) still has the integrand (x^2+y^2), but the integration now runs over the entire volume of the body. (You can think of the object as made up of many slices parallel to the x-y plane, and you just need to add

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(so $\vec{\omega} = \omega\hat{\mathbf{z}}$ here). There is no great need to use the vector $\vec{\omega}$ in the constant $\hat{\mathbf{L}}$ case in this chapter, so we won't. But don't worry, in the next chapter you'll get all the practice with $\vec{\omega}$ that you could possibly hope for.

up the I_z's of all these slices.) Hence, eq. (7.4) is the definition of I_z for *any* object (as long as the axis of rotation points in the z-direction).

The reason why the analysis in this chapter is not general is: (1) we are restricting the rotation axis to point along the z-direction, and (2) even with this restriction, a non-planar object might have non-zero x- and y-components of $\mathbf{L}$; we found only the z-component in eq. (7.5). This second fact is strange, but true. Ponder it for now; we'll deal with it in Section 8.2.

But we're getting ahead of ourselves. Until the next chapter, suffice it to say that if the axis of rotation points in the z-direction, then eqs. (7.4) and (7.5) correctly give L_z, for any object. ♣

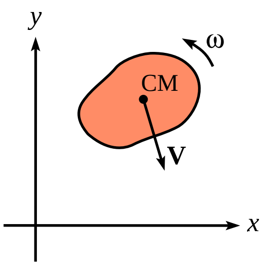

Figure 7.3

7.1.2 General motion in x-y plane

How do we deal with general motion in the x-y plane? For the motion in Fig. 7.3, the various pieces of mass are not traveling in circles about the origin, so we cannot write $v = \omega r$, as we did above.

It turns out to be highly advantageous to write the angular momentum, $\mathbf{L}$, and the kinetic energy, T, in terms of the center-of-mass (CM) coordinates and the coordinates relative to the CM. The expressions for $\mathbf{L}$ and T take on very nice forms when written this way, as we now show.

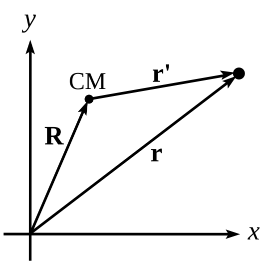

Figure 7.4

Let the coordinates of the CM be $\mathbf{R} = (X, Y)$, and let the coordinates relative to the CM be $\mathbf{r}' = (x', y')$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 7.4). Let the velocity of the CM be $\mathbf{V}$, and let the velocity relative to the CM be $\mathbf{v}'$. Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$. Let the body rotate with angular speed ω' around the CM (around an instantaneous axis parallel to the z-axis, so that the pancake remains in the x-y plane at all times).[4] Then $v' = \omega' r'$.

Let's look at $\mathbf{L}$ first. The angular momentum is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times (\mathbf{V} + \mathbf{v}')\, dm \\
&= M\mathbf{R} \times \mathbf{V} + \int \mathbf{r}' \times \mathbf{v}'\, dm \qquad \text{(cross terms vanish; see below)} \\
&= M\mathbf{R} \times \mathbf{V} + \left(\int r'^2 \omega' dm \right) \hat{\mathbf{z}} \\
&\equiv M\mathbf{R} \times \mathbf{V} + \left(I_z^{\mathrm{CM}} \omega' \right) \hat{\mathbf{z}}. \qquad (7.9)
\end{aligned}$$

where M is the mass of the pancake. In going from the second to third line above, the cross terms, $\int \mathbf{r}' \times \mathbf{V}\, dm$ and $\int \mathbf{R} \times \mathbf{v}'\, dm$, vanish because of the definition of the CM (which says that $\int \mathbf{r}'\, dm = 0$, and hence $\int \mathbf{v}'\, dm = d(\int \mathbf{r}'\, dm)/dt = 0$). The quantity I_z^{CM} is the moment of inertia around an axis through the CM, parallel to the z-axis.

[4] What we mean here is the following. Consider a coordinate system whose origin is the CM and whose axes are parallel to the fixed x- and y-axes. Then the pancake rotates with angular speed ω' in this reference frame.

Eq. (7.9) is a very nice result, and it's important enough to be called a theorem. In words, it says:

Theorem 7.1 *We can find the angular momentum (relative to the origin) of a body by treating it as a point mass located at the CM and finding the angular momentum of this point mass (relative to the origin), and by then adding on the angular momentum of the body (relative to the CM).* [5]

Note that if we have the special case where the CM travels around the origin in a circle, with angular speed Ω, then eq. (7.9) becomes $\mathbf{L} = \left(MR^2\Omega + I_z^{\rm CM}\omega'\right)\hat{\mathbf{z}}$.

Now let's look at T. The kinetic energy is

$$\begin{aligned}
T &= \int \frac{1}{2}v^2\,dm \\
&= \int \frac{1}{2}|\mathbf{V}+\mathbf{v}'|^2\,dm \\
&= \frac{1}{2}MV^2 + \int \frac{1}{2}v'^2\,dm \qquad \text{(cross term vanishes; see below)} \\
&= \frac{1}{2}MV^2 + \int \frac{1}{2}r'^2\omega'^2 dm \\
&\equiv \frac{1}{2}MV^2 + \frac{1}{2}I_z^{\rm CM}\omega'^2.
\end{aligned} \tag{7.10}$$

In going from the second to third line above, the cross term $\int \mathbf{V}\cdot\mathbf{v}'\,dm$ vanishes by definition of the CM.

Again, eq. (7.10) is a very nice result. In words, it says:

Theorem 7.2 *We can find the kinetic energy of a body by treating it as a point mass located at the CM, and by then adding on the kinetic energy of the body due to motion relative to the CM.*

7.1.3 The parallel-axis theorem

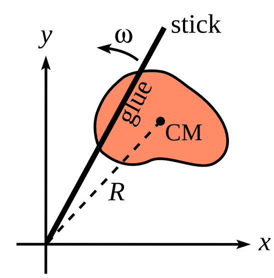

Figure 7.5

Consider the special case where the CM rotates around the origin at the same rate as the body rotates around the CM. (This may be achieved, for example, by gluing a long stick across the pancake and pivoting one end of the stick at the origin; see Fig. 7.5.) This means that we have the nice situation where all points in the pancake travel in circles around the origin. Let their angular speed be ω.

In this situation, the speed of the CM is ωR, so eq. (7.9) says that the angular momentum around the origin is

$$L_z = (MR^2 + I_z^{\rm CM})\omega. \tag{7.11}$$

[5] We could have chosen a point P other than the CM, and then written things in terms of the coordinates of P and the coordinates relative to P (which could also be described by a rotation). But then the cross terms in eq. (7.9) wouldn't vanish, and we'd end up with an unenlightening mess.

In other words, the moment of inertia around the origin is

$$\boxed{I_z = MR^2 + I_z^{\rm CM}}. \tag{7.12}$$

This is the *parallel-axis theorem*. It says that once you've calculated the moment of inertia of an object relative to the CM (namely $I_z^{\rm CM}$), then if you want to calculate I_z around an arbitrary point in the plane of the pancake, you simply have to add on MR^2, where R is the distance from the point to the CM, and M is the mass of the pancake.

The parallel-axis theorem is simply a special case of the more general result, eq. (7.9), so it is valid *only* with the CM, and not with any other point.

Likewise, in this situation, eq. (7.10) gives

$$T = \frac{1}{2}(MR^2 + I_z^{\rm CM})\omega^2. \tag{7.13}$$

Example (A stick): Let's verify the parallel-axis theorem for a stick of mass m and length ℓ, in the case where we want to find the moment of inertia about an end. (Both of the relevant axes will be perpendicular to the stick, and parallel to each other, of course.)

For convenience, let $\rho = m/\ell$ be the density. The moment of inertial about an end is

$$I^{\rm end} = \int_0^\ell x^2\, dm = \int_0^\ell x^2 \rho\, dx = \frac{1}{3}\rho\ell^3 = \frac{1}{3}(\rho\ell)\ell^2 = \frac{1}{3}m\ell^2. \tag{7.14}$$

The moment of inertial about the CM is

$$I^{\rm CM} = \int_{-\ell/2}^{\ell/2} x^2\, dm = \int_{-\ell/2}^{\ell/2} x^2 \rho\, dx = \frac{1}{12}\rho\ell^3 = \frac{1}{12}m\ell^2. \tag{7.15}$$

This is consistent with the parallel axis theorem, eq. (7.12), because

$$I^{\rm end} = m\left(\frac{\ell}{2}\right)^2 + I^{\rm CM}. \tag{7.16}$$

Remember that this only works with the CM. If you instead want to compare $I^{\rm end}$ with the I around a point, say, $\ell/6$ from that end, then you cannot say they differ by $m(\ell/6)^2$. But you *can* compare each of them to $I^{\rm CM}$ and say that they differ by $(\ell/2)^2 - (\ell/3)^2 = 5\ell^2/36$.

7.1.4 The perpendicular-axis theorem

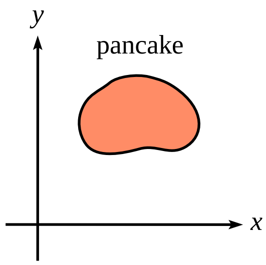

Figure 7.6

This theorem is valid *only* for pancake objects. Consider a pancake object in the x-y plane (see Fig. 7.6). Then the *perpendicular-axis theorem* says that

$$I_z = I_x + I_y, \tag{7.17}$$

where I_x and I_y are defined analogously to eq. (7.4). (That is, to find I_x, you imagine spinning the object around the x-axis at angular speed ω, and then define $I_x \equiv L_x/\omega$. Similarly for I_y.) So we have

$$I_x \equiv \int (y^2 + z^2)\, dm, \qquad I_y \equiv \int (z^2 + x^2)\, dm, \qquad I_z \equiv \int (x^2 + y^2)\, dm. \qquad (7.18)$$

To prove the theorem, we simply note that $z = 0$ for our pancake object. Hence $I_z = I_x + I_y$.

In the limited number of cases where this theorem is applicable, it may save you some trouble. A few examples are given in the next section.

7.2 Calculating moments of inertia

7.2.1 Lots of examples

Let's now compute some moments of inertia for a few objects, around specified axes. We will use ρ to denote mass density (per unit length, area, or volume, as appropriate). We will assume that this density is uniform throughout the object. For the more complicated objects, it is generally a good idea to slice the object up into pieces for which I is already known. The problem then reduces to integrating over these known I's. There is usually more than one way to do this slicing. For example, a sphere may be looked at as a series of concentric shells or a collection of disks stacked on top of each other. In the examples below, you may find other slicings more appealing than the ones given.

Consider at least a few of these examples as problems and try to work them out for yourself.

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1. A ring of mass M and radius R (axis through center, perpendicular to plane; Fig. 7.7).
   $I = \int r^2\, dm = \int_0^{2\pi} R^2 \rho R\, d\theta = R^2(2\pi R\rho) = \boxed{MR^2}$, as it should be.

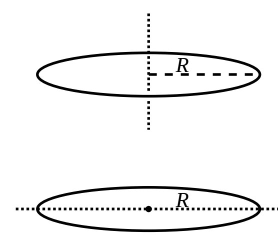


Figure 7.7

2. A ring of mass M and radius R (axis through center, in plane; Fig. 7.7).
   The distance from the axis is (the absolute value of) $R\sin\theta$. Therefore, $I = \int r^2\, dm = \int_0^{2\pi} (R\sin\theta)^2 \rho R\, d\theta = R^2(1/2)(2\pi R\rho) = \boxed{\frac{1}{2}MR^2}$.
   You can also do this via the perpendicular axis theorem. In the notation of section 7.1.4, we have $I_x = I_y$, by symmetry. Hence, $I_z = 2I_x$. Using $I_z = MR^2$ from Example 1 gives the proper result.
3. A disk of mass M and radius R (axis through center, perpendicular to plane; Fig. 7.8).
   $I = \int r^2\, dm = \int_0^{2\pi}\int_0^R r^2 \rho r\, dr\, d\theta = R^2(\pi R^2\rho/2) = \boxed{\frac{1}{2}MR^2}$.
   You can save one (trivial) integration step by considering the disk to be made up of many concentric rings, and invoke Example 1. The mass of each ring is $\rho 2\pi r\, dr$. Integrating over the rings gives $I = \int_0^R (\rho 2\pi r\, dr) r^2 = \pi R^4 \rho/2 = MR^2/2$, as before. Slicing the disk up is fairly inconsequential in this example, but it will save a good deal of effort in others.

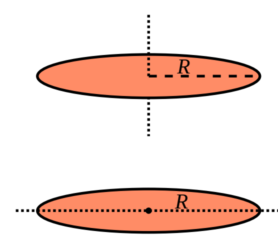


Figure 7.8

4. A disk of mass M and radius R (axis through center, in plane; Fig. 7.8).

   Slice the disk up into rings, and use Example 2. $I = \int_0^R (1/2)(\rho 2\pi r\, dr)r^2 = R^2(\pi R^2 \rho/4)$ $= \boxed{\frac{1}{4}MR^2}$.

   Or, just use Example 3 and the perpendicular axis theorem.

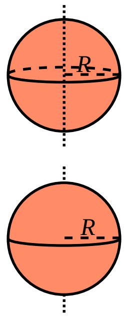


Figure 7.9

5. A spherical shell of mass M and radius R (any axis through center; Fig. 7.9).

   Choose the z-axis. We may slice the sphere into a large number of horizontal ring-like strips. In polar coords, the radii of the rings are given by $R\sin\theta$. The area of a strip is then $2\pi(R\sin\theta)R d\theta$. So we have $I = \int (x^2+y^2)\, dm = \int_0^\pi (R\sin\theta)^2 \rho(2\pi R\sin\theta)R d\theta$. Using $\int \sin^3\theta = \int \sin\theta(1-\cos^2\theta) = -\cos\theta + \cos^3\theta/3$, we obtain $I = \boxed{\frac{2}{3}MR^2}$.

6. A sphere of mass M and radius R (any axis through center; Fig. 7.9).

   A sphere is made up of concentric spherical shells. The volume of a shell is $4\pi r^2 dr$. Using Example 5, we have $I = \int_0^R (2/3)(4\pi\rho r^2 dr)r^2 = (2/5)(4/3\pi R^5 \rho) = \boxed{\frac{2}{5}MR^2}$.

   It's a good exercise to do this one from scratch, using polar coords.

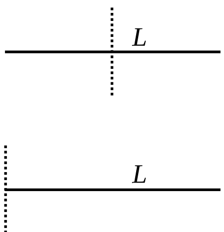


Figure 7.10

7. A thin uniform rod of mass M and length L (axis through center, perpendicular to rod; Fig. 7.10).

   $I = \int x^2 dm = \int_{-L/2}^{L/2} x^2 \rho\, dx = (1/12)L^2(\rho L) = \boxed{\frac{1}{12}ML^2}$.

8. A thin uniform rod of mass M and length L (axis through end, perpendicular to rod; Fig. 7.10).

   $I = \int x^2 dm = \int_0^L x^2 \rho\, dx = (1/3)L^2(\rho L) = \boxed{\frac{1}{3}ML^2}$.

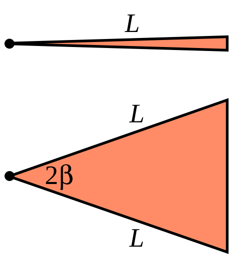


Figure 7.11

9. An infinitesimally thin triangle of mass M and length L (axis through tip, perpendicular to plane; Fig. 7.11).

   Let the base have length a (we will assume a is infinitesimally small). Then a slice at a distance x from the tip has length $a(x/L)$. If the slice has thickness dx, then it is essentially a point mass of mass $dm = \rho a x\, dx/L$. So $I = \int x^2 dm = \int_0^L x^2 \rho a x/L\, dx =$ $(1/2)L^2(\rho aL/2) = \boxed{\frac{1}{2}ML^2}$ (since $aL/2$ is the area of the triangle). This of course has the same form as the disk in Example 3, because a disc is made up of many of these triangles.

10. An isosceles triangle of mass M, vertex angle 2β, and common-side length L (axis through tip, perpendicular to plane; Fig. 7.11).

    Let h be the altitude of the triangle (so $h = L\cos\beta$). Slice the triangle up as in Fig. 7.11. Let y be the coordinate along the base of the triangle (so $-h\tan\beta \le y \le h\tan\beta$). With dy the base of a thin triangle, and h the altitude, the area of a thin triangle is $h\, dy/2$. And its length is $\sqrt{h^2+y^2}$.

    So from Example 9, we have $I = \int_{-h\tan\beta}^{h\tan\beta} \rho(h\, dy/2)(h^2+y^2)/2 = (\rho h^4/2)(\tan\beta + \tan^3\beta/3)$. But the area of the whole triangle is $h^2\tan\beta$, so we have $I = (Mh^2/2)(1+\tan^2\beta/3)$. In terms of L, this is $I = (ML^2/2)(\cos^2\beta + \sin^2\beta/3) = \boxed{\frac{1}{2}ML^2(1-\frac{2}{3}\sin^2\beta)}$.

    You can also slice the triangle into strips parallel to the base.

11. A regular N-gon of mass M and 'radius' R (axis through center, perpendicular to plane; Fig. 7.12).

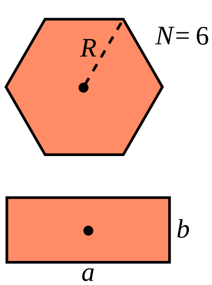


Figure 7.12

The N-gon is made up of N isosceles triangles, so we can use Example 10, with $\beta = \pi/N$. The masses of the triangles simply add, so if M is the mass of the whole N-gon, we have $I = \boxed{\frac{1}{2}MR^2(1 - \frac{2}{3}\sin^2\frac{\pi}{N})}$.

Let's list the values of I for a few N. We'll use the shorthand notation $(N, I/MR^2)$. We have $(3, \frac{1}{4})$, $(4, \frac{1}{3})$, $(6, \frac{5}{12})$, $(\infty, \frac{1}{2})$. These values of I form a nice arithmetic progression.

12. A rectangle of mass M and sides of length a and b (axis through center, perpendicular to plane; Fig. 7.12).

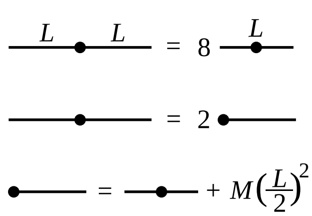
    Let the z-axis be perpendicular to the plane. We know that $I_x = Mb^2/12$ and $I_y = Ma^2/12$, so the perpendicular axis theorem tells us that $I_z = I_x + I_y = \boxed{\frac{1}{12}M(a^2 + b^2)}$.

7.2.2 A neat trick

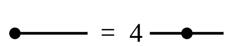
For some objects with the proper symmetry, it is possible to calculate I without doing any integrals. All that is needed is a scaling argument and the parallel-axis theorem. We will illustrate this technique by finding I for a stick (Example 7, above). Other applications can be found in Problems 4 and 5.

In the present example, the basic trick is to compare I for a stick of length L with I for a stick of length $2L$. A simple scaling argument shows the latter is eight times the former. This follows because the integral $\int x^2\,dm = \int x^2\rho\,dx$ has three powers of x in it. So a change of variables $y = 2x$ brings in a factor of $2^3 = 8$. In other words, if we imagine expanding the smaller stick to make the larger one, then a corresponding piece will now be twice as far from the axis, and also twice as massive.

This problem is most easily done with pictures. If we denote a moment of inertia of an object by a picture of the object (with a dot signifying the axis), then we have:



The first line comes from the scaling argument, the second is obvious (moments of inertia simply add, and the left-hand side is two copies of the right-hand side, attached at the pivot), and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two gives



Plugging this expression for into the third equation gives

 $= \frac{1}{12}ML^2$

Note that sooner or later you must use real live numbers (which enter here through the parallel axis theorem). Using nothing but scaling arguments isn't enough, because they provide only linear equations homogeneous in the I's, and therefore give no way to pick up the proper dimensions.

Once you've mastered this trick and applied it to the fractal objects in Problem 5, you can impress your friends by saying that you can "use scaling arguments, along with the parallel-axis theorem, to calculate moments of inertia of objects with fractal dimension." (And you never know when that might come in handy.)

7.3 Torque

We will now show that (under certain conditions, stated below) the rate of change of angular momentum is equal to a certain quantity, $\boldsymbol{\tau}$, which we call the *torque*. That is, $\boldsymbol{\tau} = d\mathbf{L}/dt$. This is the rotational analog of our old friend $\mathbf{F} = d\mathbf{p}/dt$ involving linear momentum. The basic idea here is straightforward, but there are a couple subtle issues. One deals with internal forces within a collection of particles. The other concerns an origin (that is, the point relative to which the angular momentum is calculated) that is not fixed. To keep things straight, we'll prove the general theorem by dealing with three increasingly complicated situations.

Our derivation of $\boldsymbol{\tau} = d\mathbf{L}/dt$ here holds for completely general motion; we can take the result and use it in the following chapter, too. If you wish, you can construct another proof of $\boldsymbol{\tau} = d\mathbf{L}/dt$ for the special case of a pancake object in the x-y plane. But since the general proof is no more difficult, we'll present it here in this chapter and get it over with.

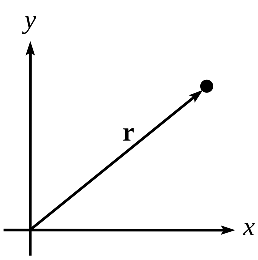


Figure 7.13

7.3.1 Point mass, fixed origin

Consider a point mass at position $\mathbf{r}$ relative to a fixed origin (see Fig. 7.13). The time derivative of the angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \mathbf{F} \\ &= 0 + \mathbf{r} \times \mathbf{F}, \end{aligned} \tag{7.19}$$

where $\mathbf{F}$ is the force acting on the particle. (This is the same proof as in Section 6.1, except that here we are considering an arbitrary force instead of a central one.) Therefore, if we define the *torque* on the particle as

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{7.20}$$

then we have

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \tag{7.21}$$

7.3.2 Extended mass, fixed origin

In an extended object, there are internal forces acting on the various pieces of the object, in addition to whatever external forces exist. For example, the external force on a given atom in a body might come from gravity, while the internal forces come from the adjacent atoms. How do we deal with these different types of forces?

In what follows, we will deal only with internal forces that are central forces, that is, where the force between two objects is directed along the line joining them. This is a valid assumption for the pushing and pulling forces between atoms in a solid. (It isn't valid, for example, when dealing with magnetic forces. But we won't be interested with such things here.) We will invoke Newton's third law, which says that the force that particle 1 applies to particle 2 is equal and opposite to the force that particle 2 applies to particle 1.

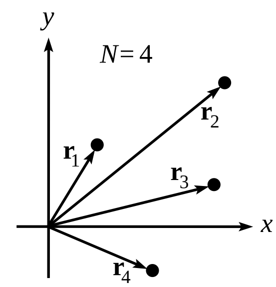


Figure 7.14

For concreteness, let us assume that we have a collection of N discrete particles labeled by the index i (see Fig. 7.14). (In the continuous case, we'd need to replace the following sums with integrals.) Then the total angular momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i. \tag{7.22}$$

The force acting on each particle is $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} = d\mathbf{p}_i/dt$. Therefore,

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i \\ &= \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \\ &= \sum_i \mathbf{v}_i \times (m\mathbf{v}_i) + \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}) \\ &= 0 + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \sum_i \boldsymbol{\tau}_i^{\text{ext}}. \end{aligned} \tag{7.23}$$

The last line follows because $\mathbf{v}_i \times \mathbf{v}_i = 0$, and also $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$, as you can show in Problem 15. (This is fairly obvious. It basically says that a rigid object with no external forces won't spontaneously start rotating.) Note that the right-hand side involves the *total* torque acting on the body, which may come from forces acting at many different points.

7.3.3 Extended mass, non-fixed origin

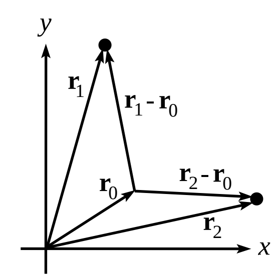


Figure 7.15

Let the position of the origin be $\mathbf{r}_0$ (see Fig. 7.15). Let the positions of the particles be $\mathbf{r}_i$ ($\mathbf{r}_0$, $\mathbf{r}$, and all other vectors below are measured with respect to a given fixed coordinate system). Then the total angular momentum of the system, relative to the (possibly moving) origin $\mathbf{r}_0$, is

$$\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0). \tag{7.24}$$

Therefore,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)\right) \\ &= \sum_i m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) + \sum_i m_i(\mathbf{r}_i - \mathbf{r}_0) \times (\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_0) \\ &= 0 + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} - m_i\ddot{\mathbf{r}}_0), \end{aligned} \tag{7.25}$$

because $m_i\ddot{\mathbf{r}}_i$ is the net force (namely $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$) acting on the ith particle. But the method of Problem 15 shows that the term involving $\mathbf{F}_i^{\text{int}}$ vanishes (prove this). And since $\sum m_i\mathbf{r}_i = M\mathbf{R}$ (where $M = \sum m_i$ is the total mass, and $\mathbf{R}$ is the position of the center-of-mass), we have

$$\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} - M(\mathbf{R} - \mathbf{r}_0) \times \ddot{\mathbf{r}}_0. \tag{7.26}$$

The first term is the external torque, relative to the origin $\mathbf{r}_0$. The second term is something we wish would go away. And indeed, it usually does. It vanishes if any of the following three conditions is satisfied.

1. $\mathbf{R} = \mathbf{r}_0$. That is, the origin is the CM.
2. $\ddot{\mathbf{r}}_0 = 0$. That is, the origin is not accelerating.
3. $(\mathbf{R} - \mathbf{r}_0)$ is parallel to $\ddot{\mathbf{r}}_0$. This condition is rarely invoked.

If any of these conditions is satisfied, then we are free to write

$$\boxed{\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} = \sum_i \boldsymbol{\tau}_i^{\text{ext}}}, \tag{7.27}$$

that is, we can equate the total torque with the rate of change of the total angular momentum. An immediate corollary of this result is:

Corollary 7.3 *If the total torque on a system is zero, then its angular momentum is conserved. In particular, the angular momentum of an isolated system (one that is subject to no external forces) is conserved.*

In the present chapter, we are dealing only with cases where $\hat{\mathbf{L}}$ is constant. Therefore, $d\mathbf{L}/dt = (dL/dt)\hat{\mathbf{L}}$. But $L = I\omega$, so $dL/dt = I\dot{\omega} \equiv I\alpha$. Taking the magnitude of each side of eq. (7.27) therefore gives

$$\tau = I\alpha, \tag{7.28}$$

if $\hat{\mathbf{L}}$ is constant.

Invariably, we will calculate angular momentum and torque around either a fixed point or the CM. These are "safe" origins, in the sense that eq. (7.27) holds. As long as you vow to always use one of these safe origins, you can simply apply eq. (7.27) and basically ignore most of its derivation.

REMARK: There is one common situation where the third condition above is relevant. Consider a circle rolling on the ground. The instantaneous point of contact on the circle is a valid choice for the origin. This is true because $(\mathbf{R} - \mathbf{r}_0)$ points vertically. And $\ddot{\mathbf{r}}_0$ also points vertically. (A point on a rolling circle traces out a cycloid. Right before the point hits the ground, it is moving straight downward; right after it hits the ground, it is moving straight upward.) ♣

7.4 Impulse

In Section 4.5, we defined the *impulse*, $\mathcal{I}$, applied to an object to be the time integral of the force applied (which is the net change in linear momentum). That is,

$$\mathcal{I} \equiv \int_{t_1}^{t_2} \mathbf{F}(t)\, dt = \Delta \mathbf{p}. \tag{7.29}$$

We now define the *angular impulse*, $\mathcal{I}_\theta$, applied to an object to be the time integral of the torque applied (which is the net change in angular momentum). That is,

$$\mathcal{I}_\theta \equiv \int_{t_1}^{t_2} \boldsymbol{\tau}(t)\, dt = \Delta \mathbf{L}. \tag{7.30}$$

These are just definitions, devoid of any content. The place where the physics comes in is the following. Consider a situation where $\mathbf{F}(t)$ is always applied at the same position relative to the origin around which $\boldsymbol{\tau}(t)$ is calculated. Let this position be $\mathbf{R}$. Then we have $\boldsymbol{\tau}(t) = \mathbf{R} \times \mathbf{F}(t)$. Plugging this into eq. (7.30), and taking the constant $\mathbf{R}$ outside the integral, gives $\mathcal{I}_\theta = \mathbf{R} \times \mathcal{I}$. That is,

$$\Delta \mathbf{L} = \mathbf{R} \times (\Delta \mathbf{p}) \qquad (\text{for } \mathbf{F}(t) \text{ applied at one position}). \tag{7.31}$$

This is a very useful result. It deals with the net changes in $\mathbf{L}$ and $\mathbf{p}$, and not with their changes at any particular instant. Hence, even if the magnitude of $\mathbf{F}$ is changing in some arbitrary manner as time goes by, and we have no idea what $\Delta\mathbf{p}$ and $\Delta\mathbf{L}$ are, eq. (7.31) is still true. (Eq. (7.31) holds for general motion, so we can apply it in the next chapter, too.)

The prime example of an impulse is the striking of a stick with a hammer. Let $\mathbf{F}(t)$ be perpendicular to a stick, at its end. Let the stick have mass m and length ℓ. Let the blow occur quickly, so that the stick doesn't move much while the hammer is in contact. We have no idea exactly what $\mathbf{F}(t)$ looks like, or for how long it is applied, but we do know from eq. (7.31) that $\Delta L = (\ell/2)\Delta p$ (where L is calculated relative to the CM). If the stick was initially at rest, this implies $(m\ell^2/12)\omega = (\ell/2)mv$. Therefore, the final v and ω are related by $\ell\omega = 6v$, independent of what is going on during the blow.

Impulse is also useful for 'collisions' that occur over extended times (see for example, Problem 17).

7.5 Exercises

Section 7.4: Impulse

1. **Repetitive bouncing** $*$

   Using the result of Problem 18, what must the relation be between v_x and ω, so that a superball will continually bounce back and forth between the same two points of contact on the ground?

2. **Bouncing under a table** $*$

   You throw a superball so that it bounces off the floor, then off the underside of a table, then off the floor again. What must the initial relation between v_x and ω be, so that the ball returns to your hand?[6] (Use the result of Problem 18, and modifications thereof.)

[6] You are strongly encouraged to bounce a ball in such a manner and have it magically come back to your hand. It turns out that the required value of ω is rather small, so a natural throw with $\omega \approx 0$ will essentially get the job done (as you will discover).

7.6 Problems

Section 7.1: Pancake object in x-y plane

1. **Leaning rectangle** ***

   A rectangle of height $2a$ and width $2b$ rests on top of a fixed cylinder of radius R (see Fig. 7.16). The moment of inertia of the rectangle around its center is I. The rectangle is given an infinitesimal kick, and then 'rolls' on the cylinder without slipping. Find the equation of motion for the tilting angle of the rectangle. Under what conditions will it fall off the cylinder, and under what conditions will oscillate back and forth? Find the frequency of these small oscillations.

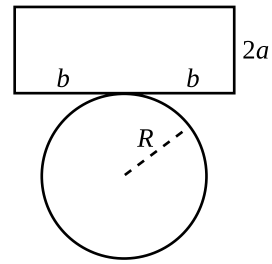


Figure 7.16

2. **Leaving the sphere** **

   (a) A small particle rests on top of a frictionless sphere. The particle is given an infinitesimal kick and slides downward (see Fig. 7.17). At what point does it lose contact with the sphere?

   (b) A small ball with moment of inertia $\eta m r^2$ rests on top of a sphere. There is friction between the ball and sphere. The ball is given an infinitesimal kick and rolls downward without slipping (see Fig. 7.17). At what point does it lose contact with the sphere? (Assume that r is much less than the radius of the sphere.)

   How does your answer change if the size of the ball is comparable to, or larger than, the size of the sphere? (Assume that the sphere is fixed.)

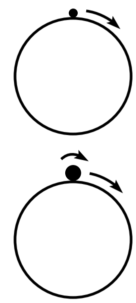


Figure 7.17

3. **Sliding ladder** ***

   A ladder of length ℓ and uniform mass density per unit length leans against a frictionless wall. The ground is also frictionless. The ladder is initially held motionless, with its bottom end an infinitesimal distance from the wall. The ladder is then released, whereupon the bottom end slides away from the wall, and the top end slides down the wall (see Fig. 7.18).

   A long time after the ladder is released, what is the horizontal component of the velocity of its center of mass?

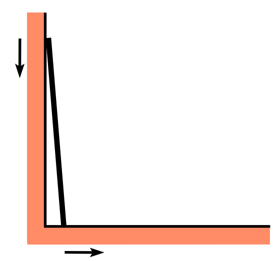


Figure 7.18

Section 7.2: Calculating moments of inertia

4. **Slick calculations of I** **

   In the spirit of section 7.2.2, find the moments of inertia of the following objects (see Fig. 7.19).

   (a) A uniform square of mass m and side ℓ (axis through center, perpendicular to plane).

   (b) A uniform equilateral triangle of mass m and side ℓ (axis through center, perpendicular to plane).

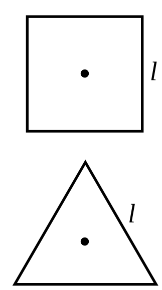


Figure 7.19

5. **Slick calculations of I for fractal objects** ***

In the spirit of section 7.2.2, find the moments of inertia of the following fractal objects. (Be careful how the mass scales.)

(a) Take a stick of length ℓ, and remove the middle third. Then remove the middle third from each of the remaining two pieces. Then remove the middle third from each of the remaining four pieces, and so on, forever. Let the final object have mass m (axis through center, perpendicular to stick; see Fig. 7.20).[7]

(b) Take a square of side ℓ, and remove the 'middle' square (1/9 of the area). Then remove the 'middle' square from each of the remaining eight squares, and so on, forever. Let the final object have mass m (axis through center, perpendicular to plane; see Fig. 7.21).

(c) Take an equilateral triangle of side ℓ, and remove the 'middle' triangle (1/4 of the area). Then remove the 'middle' triangle from each of the remaining three triangles, and so on, forever. Let the final object have mass m (axis through center, perpendicular to plane; Fig. 7.22).

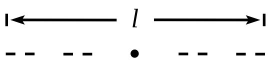


Figure 7.20

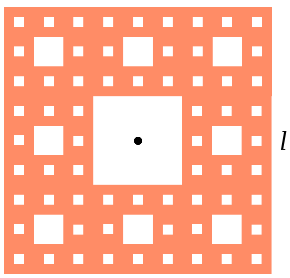


Figure 7.21

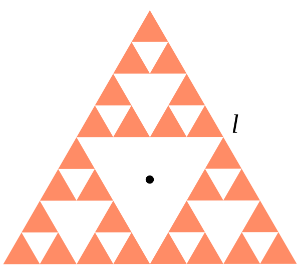


Figure 7.22

6. **Minimum I**

A moldable blob of matter of mass M is to be situated between the planes $z = 0$ and $z = 1$ (see Fig. 7.23). The goal is to have the moment of inertia around the z-axis be as small as possible. What shape should the blob take?

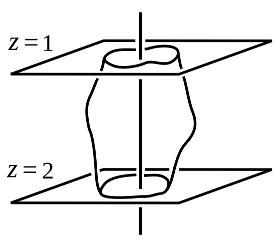


Figure 7.23

Section 7.3: Torque

7. **Removing a support**

(a) A rod of length ℓ and mass m rests on supports at its ends. The right support is quickly removed (see Fig. 7.24). What is the force on the left support immediately thereafter?

(b) A rod of length $2r$ and moment of inertia $\eta m r^2$ (where η is a numerical constant) rests on top of two supports, each of which is a distance d away from the center. The right support is quickly removed (see Fig. 7.24). What is the force on the left support immediately thereafter?

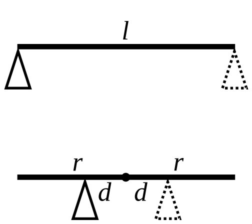


Figure 7.24

[7] This object is the Cantor set, for those who like such things. It has no length, so the density of the remaining mass is infinite. If you suddenly develop an aversion to point masses with infinite density, simply imagine the above iteration being carried out only, say, a million times.

8. **Oscillating ball** *

A small ball (with uniform density) of radius r rolls without slipping near the bottom of a fixed cylinder of radius R (see Fig. 7.25). What is the frequency of small oscillations about the bottom? (Assume $r \ll R$.)

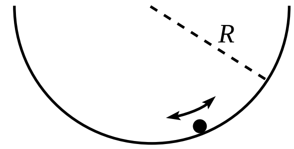


Figure 7.25

9. **Ball hitting stick** *

A ball of mass M hits a stick with moment of inertia $I = \eta m\ell^2$. The ball is initially traveling with velocity V_0, perpendicular to the stick. The ball strikes the stick at a distance d from its center (see Fig. 7.26). The collision is elastic.

Find the resulting translational and rotational speeds of the stick, and also the resulting speed of the ball.

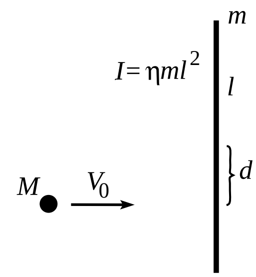


Figure 7.26

10. **A ball and stick theorem** *

A ball of mass M hits a stick with moment of inertia I. The ball is initially traveling with velocity V_0, perpendicular to the stick. The ball strikes the stick at a distance d from its center (see Fig. 7.26). The collision is elastic.

Prove that the relative speed of the ball and the point of contact on the stick is the same before and immediately after the collision. (This theorem is analogous to the 'relative speed' theorem of two balls.)

11. **A triangle of circles** ***

Three circular objects with moments of inertia $I = \eta MR^2$ are situated in a triangle as in Fig. 7.27. Find the initial downward acceleration of the top circle, if:

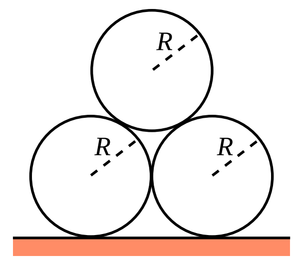


Figure 7.27

(a) There is friction between the bottom two circles and the ground (so they roll without slipping), but there is no friction between any of the circles.

(b) There is no friction between the bottom two circles and the ground, but there is friction between the circles.

Which case has a larger acceleration?

12. **Lots of sticks** ***

This problem deals with rigid 'stick-like' objects of length $2r$, masses M_i, and moments of inertia $\eta M_i r^2$. The center-of-mass of each stick is located at the center of the stick. (All the sticks have the same r and η. Only the masses differ.) Assume $M_1 \gg M_2 \gg M_3 \gg \cdots$.

The sticks are placed on a horizontal frictionless surface, as shown in Fig. 7.28. The ends overlap a negligible distance, and the ends are a negligible distance apart.

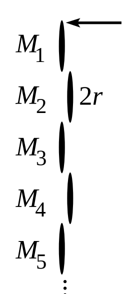


Figure 7.28

The first (heaviest) stick is given an instantaneous blow (as shown) which causes it to translate and rotate. (The blow comes from the side of stick #1 on which stick #2 lies [the right side, as shown in the figure].) The first stick will strike the second stick, which will then strike the third stick, and so on. Assume all collisions among the sticks are elastic.

Depending on the size of η, the speed of the nth stick will either (1) approach zero, (2) approach infinity, or (3) be independent of n, as $n \to \infty$.

What is the special value of η corresponding to the third of these three scenarios? Give an example of a stick having this value of η.

(You may work in the approximation where M_1 is infinitely heavier than M_2, which is infinitely heavier than M_3, etc.)

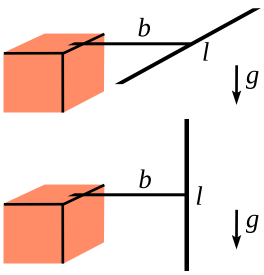


Figure 7.29

13. **Falling stick** $*$

A massless stick of length b has one end attached to a pivot and the other end glued perpendicularly to the middle of a stick of mass m and length ℓ.

(a) If the two sticks are held in a horizontal plane (see Fig. 7.29) and then released, what is the initial acceleration of the CM?

(b) If the two sticks are held in a vertical plane (see Fig. 7.29) and then released, what is the initial acceleration of the CM?

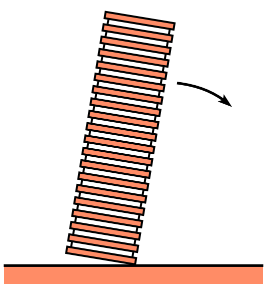


Figure 7.30

14. **Falling Chimney** $****$

A chimney initially stands upright. It is given a tiny kick, so that it topples over. At what point along its length is it most likely to break?

In doing this problem, work with the following two-dimensional simplified model of a chimney. Assume the chimney consists of boards stacked on top of each other; and each board is attached to the two adjacent ones with strings at each end (see Fig. 7.30). Assume that the boards are slightly thicker at their ends, so that they only touch each other at their endpoints. The goal is to find where the string has the maximum tension.

(In doing this problem, you may work in the approximation where the width of the chimney is very small compared to its height.)

15. **Zero torque from internal forces** $**$

Given a collection of particles with positions $\mathbf{r}_i$, let the force on the ith particle, due to all the others, be $\mathbf{F}_i^{\text{int}}$. Assuming that the force between any two particles is a central force, use Newton's third law to show $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$.

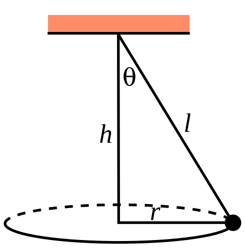


Figure 7.31

16. **Lengthening the string** $**$

A mass hangs from a string and swings around in a circle, as shown in Fig. 7.31. The length of the string is very slowly increased (or decreased). Let θ, ℓ, r, and h be defined as in the figure.

(a) Assuming θ is very small, how does r depend on ℓ?

(b) Assuming θ is very close to $\pi/2$, how does h depend on ℓ?

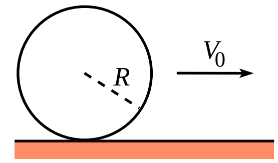


Figure 7.32

Section 7.4: Impulse

17. **Sliding to rolling** $**$

A ball initially slides, without rotating, on a horizontal surface with friction (see Fig. 7.32). The initial speed of the ball is V_0, and the moment of inertia about its center is $I = \eta m R^2$.

(a) Without knowing anything about how the friction force depends on position, find the speed of the ball when it begins to roll without slipping. Also, find the kinetic energy lost while sliding.

(b) Now consider the special case where the coefficient of sliding friction is μ, independent of position. At what time, and at what distance, does the ball begin to roll without slipping?
Verify that the work done by friction equals the loss in energy calculated in part (a) (be careful on this).

18. **The superball** $**$

A ball with radius R is thrown in the plane of the paper (the x-y plane), while also spinning around the axis perpendicular to the page. The ball bounces off the floor. Assuming that the collision is elastic, show that the v_x' and ω' after the bounce are related to the v_x and ω before the bounce by

$$\begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix} \begin{pmatrix} v_x \\ R\omega \end{pmatrix}, \tag{7.32}$$

where our convention is that positive v_x is to the right, and positive ω is clockwise.[8]

19. **Many bounces** $*$

Using the result of Problem 18, describe what happens over the course of many superball bounces.

20. **Rolling over a bump** $**$

A ball with radius R (and uniform density) rolls without slipping on the ground. It encounters a step of height h and rolls up over it. Assume that the ball sticks to the corner of the step briefly (until the center of the ball is directly above the corner). And assume that the ball does not slip with respect to the corner.

Show that the minimum initial speed, V_0, required for the ball to climb over the step, is given by

$$V_0 \geq \frac{R\sqrt{14gh/5}}{7R/5 - h}. \tag{7.33}$$

[8] Assume that there is no distortion in the ball during the bounce, which means that the forces in the x- and y-directions are independent, which then means that the kinetic energies associated with the x- and y-motions are separately conserved.

7.7 Solutions

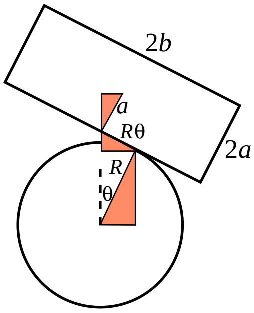


Figure 7.33

1. **Leaning rectangle**

When the rectangle has rotated through an angle θ, the position of its CM is (relative to the center of the cylinder)

$$(x, y) = R(\sin\theta, \cos\theta) + R\theta(-\cos\theta, \sin\theta) + a(\sin\theta, \cos\theta), \tag{7.34}$$

where we have added up the distances along the three shaded triangles in Fig. 7.33 (note that the contact point has moved a distance $R\theta$ along the rectangle).

We'll use the Lagrangian method to find the equation of motion and the frequency of small oscillations. Using eq. (7.34), the square of the speed of the CM is

$$v^2 = \dot{x}^2 + \dot{y}^2 = (a^2 + R^2\theta^2)\dot{\theta}^2. \tag{7.35}$$

(There's an easy way to see this clean result. The CM instantaneously rotates around the contact point with angular speed $\dot{\theta}$, and from Fig. 7.33 the distance to the contact point is $\sqrt{a^2 + R^2\theta^2}$.)

The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}M(a^2 + R^2\theta^2)\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg\Big((R + a)\cos\theta + R\theta\sin\theta\Big). \tag{7.36}$$

The equation of motion is

$$(Ma^2 + MR^2\theta^2 + I)\ddot{\theta} + MR^2\theta\dot{\theta}^2 = Mga\sin\theta - MgR\theta\cos\theta. \tag{7.37}$$

Consider small oscillations. Using the small-angle approximations, and keeping terms only to first order in θ, we obtain

$$(Ma^2 + I)\ddot{\theta} + Mg(R - a)\theta = 0. \tag{7.38}$$

Therefore, oscillatory motion occurs for $a < R$ (note that this is independent of b). The frequency of small oscillations is

$$\omega = \sqrt{\frac{Mg(R - a)}{Ma^2 + I}}. \tag{7.39}$$

Some special cases: If $I = 0$ (i.e., all the mass is located at the CM), we have $\omega = \sqrt{g(R - a)/a^2}$. If the rectangle is a uniform horizontal stick, so that $a \ll R$, $a \ll b$, and $I \approx Mb^2/3$, we have $\omega \approx \sqrt{3gR/b^2}$. If the rectangle is a vertical stick (satisfying $a < R$), so that $b \ll a$ and $I \approx Ma^2/3$, we have $\omega \approx \sqrt{3g(R - a)/4a^2}$. If in addition $a \ll R$, then $\omega \approx \sqrt{3gR/4a^2}$.

REMARKS:

(a) Without doing much work, there are two ways that we can determine the condition under which there is oscillatory motion. The first is to look at the height of the CM. Using small-angle approximations in eq. (7.34), the height of the CM is $y \approx (R + a) + (R - a)\theta^2/2$. Therefore, if $a < R$, the potential increases with θ, so the rectangle wants to decrease its θ and fall back down to the middle. If $a > R$, the potential decreases with θ, so the rectangle wants to keep increasing its θ, and thus falls off the cylinder.

The second way is to look at the horizontal positions of the CM and the contact point. Small-angle approximations in eq. (7.34) show that the former equals $a\theta$ and the latter equals $R\theta$. Therefore, if $a < R$ then the CM is to the left of the contact point, so the torque from gravity makes θ decrease, and the motion is stable. If $a > R$ then the torque from gravity makes θ increase, and the motion is unstable.

(b) The small-angle equation of motion, eq. (7.38), can also be derived using $\tau = dL/dt$, using the instantaneous contact point, P, on the rectangle as the origin around which we calculate τ and L. (From section (7.3.3), we know that it is legal to use this point when $\theta = 0$.)

However, point P cannot be used as the origin to use $\tau = dL/dt$ to calculate the exact equation of motion, eq. (7.37), because for $\theta \neq 0$ the third condition in section (7.3.3) does not hold.

It is possible to use the CM as the origin for $\tau = dL/dt$, but the calculation is rather messy. ♣

2. **Leaving the sphere**

(a) **First Solution:** Let R be the radius of the sphere, and let θ be the angle of the ball from the top of the sphere. The particle loses contact with the sphere when the normal force becomes zero, i.e., when the normal component of gravity is not large enough to account for the centripetal acceleration of the ball. Thus, the normal force becomes zero when

$$\frac{mv^2}{R} = mg\cos\theta. \tag{7.40}$$

By conservation of energy, we have $mv^2/2 = mgR(1 - \cos\theta)$. Hence $v = \sqrt{2gR(1-\cos\theta)}$. Plugging this into eq. (7.40), we see that the particle leaves the sphere when

$$\cos\theta = \frac{2}{3}. \tag{7.41}$$

This corresponds to $\theta \approx 48.2°$.

Second Solution: Assume that the particle always stays in contact with the sphere, and find the point where the horizontal component of v starts to decrease (which it of course can't do). From above, the horizontal component, v_x, is $v_x = v\cos\theta = \sqrt{2gR(1-\cos\theta)}\,\cos\theta$. The derivative of this equals 0 when $\cos\theta = 2/3$, so this is where v_x starts to decrease (if the particle stays on the sphere). Since there is no force available to make v_x decrease, contact is lost when $\cos\theta = 2/3$.

(b) In this situation, the ball still leaves the sphere when the normal force becomes zero; so eq. (7.40) is still applicable. The only change comes in the calculation of v. The ball has rotational energy, so conservation of energy gives $mgR(1 - \cos\theta) = mv^2/2 + I\omega^2/2 = mv^2/2 + \eta mr^2\omega^2/2$. Using $r\omega = v$, we have $v = \sqrt{2gR(1-\cos\theta)/(1+\eta)}$. Plugging this into eq. (7.40), we see that the ball leaves the sphere when

$$\cos\theta = \frac{2}{3+\eta}. \tag{7.42}$$

For, $\eta = 0$, this is $2/3$, of course. For a uniform ball with $\eta = 2/5$, we have $\cos\theta = 10/17$, so $\theta \approx 54°$. For, $\eta \to \infty$, we have $\cos\theta \to 0$, so $\theta \approx 90°$ (v will be very small, because most of the energy will take the form of rotational energy.)

If the size of the ball is comparable to, or bigger than, the size of the sphere, we have to take into account the fact that the CM of the ball does not move along a circle of radius R. It moves along a circle of radius $R + r$. So eq. (7.40) becomes

$$\frac{mv^2}{R+r} = mg\cos\theta. \tag{7.43}$$

Also, the conservation-of-energy equation takes the form $mg(R+r)(1-\cos\theta) = mv^2/2 + \eta mr^2\omega^2/2$. But $r\omega$ still equals v (prove this). So we have the same equations as above, except that R is replaced everywhere by $R+r$. But R didn't appear in the original answer, so the answer is unchanged.

REMARK: The method of the second solution in part (a) will *not* work here in part (b), because there *is* a force available to make v_x decrease, namely the friction force. And indeed, v_x does decrease before the rolling ball leaves the sphere. (The v in part (b) is simply $1/\sqrt{1+\eta}$ times the v in part (a), so the maximum v_x is still achieved at $\cos\theta = 2/3$, and the angle in eq. (7.42) is larger than this.) ♣

3. **Sliding ladder**

The key to this problem is the fact that the ladder will lose contact with the wall before it hits the ground. The first thing we must do is calculate exactly where this loss of contact occurs.

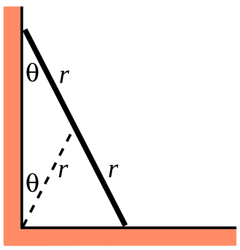


Figure 7.34

Let $r = \ell/2$, for convenience. It is easy to see that while the ladder is in contact with the wall, the CM of the ladder will move in a circle of radius r. (The median to the hypotenuse of a right triangle has half the length of the hypotenuse.) Let θ be the angle between the wall and the radius from the corner to the CM of the ladder; see Fig. 7.34. (This is also the angle between the ladder and the wall.)

We will solve the problem by assuming that the CM always moves in a circle, and then determining the point at which the horizontal CM speed starts to decrease (i.e., the point at which the normal force from the wall becomes negative, which it of course can't do).

By conservation of energy, the kinetic energy of the ladder is equal to the loss in potential energy, which is $mgr(1-\cos\theta)$, where θ is defined above. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (since the CM travels in a circle). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because θ is the angle between the ladder and the vertical.) Letting $I \equiv \eta mr^2$, to be general ($\eta = 1/3$ for our ladder), we have, by conservation of energy, $(1+\eta)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, $v = r\dot{\theta}$, is

$$v = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\,. \tag{7.44}$$

The horizontal speed is therefore

$$v_x = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{7.45}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the speed is maximum at $\cos\theta = 2/3$. (This is independent of η.)

Therefore the ladder loses contact with the wall when

$$\cos\theta = 2/3. \tag{7.46}$$

Using this value of θ in eq. (7.45) gives a horizontal speed of (letting $\eta = 1/3$)

$$v_x = \frac{\sqrt{2gr}}{3} \equiv \frac{\sqrt{g\ell}}{3}. \tag{7.47}$$

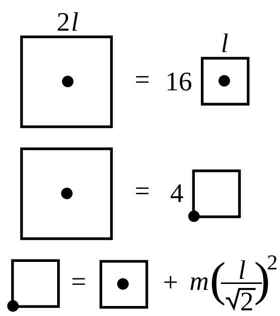
This is the horizontal speed just after the ladder loses contact with the wall, and thus is the horizontal speed from then on, because the floor exerts no horizontal force.

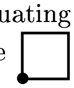
You are encouraged to compare various aspects of this problem with those in the two parts of Problem 2.

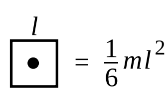
4. **Slick calculations of I**

   (a) We claim that the I for a square of side 2ℓ is 16 times the I for a square of side ℓ (where the axes pass through any two corresponding points). The factor of 16 comes in part from the fact that dm goes like the area, which is proportional to length squared. So the corresponding dm's are increased by a factor of 4. There are therefore four powers of 2 in the integral $\int r^2\, dm = \int r^2\, dx\, dy$.

   With pictures, we have:



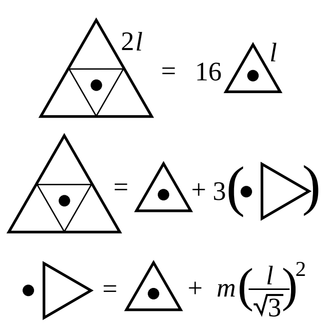
   The first line comes from the scaling argument, the second is obvious (moments of inertia add), and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

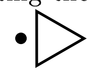


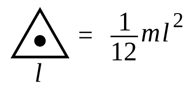
   This agrees with the result of example 12 in section 7.2.1, with $a = b = \ell$.

   (b) This is again a two-dimensional object, so the I for a triangle of side 2ℓ is 16 times the I for a triangle of side ℓ (where the axes pass through any two corresponding points).

   Again, with pictures, we have:



The first line comes from the scaling argument, the second is obvious, and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

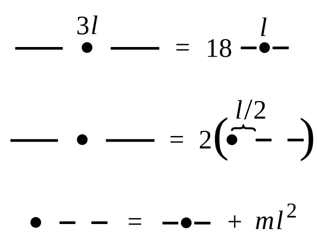


This agrees with the result of example 11 in section 7.2.1, with $N = 3$ (because the 'radius', R, used in that example equals $\ell/\sqrt{3}$).

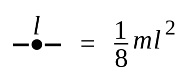
5. **Slick calculations of I for fractal objects**

(a) The scaling argument here is a little trickier than that in section 7.2.2. Our object is self-similar to an object 3 times as big, so let's increase the length by a factor of 3 and see what happens to I. In the integral $\int x^2 dm$, the x's pick up a factor of 3, so this gives a factor of 9. But what happens to the dm? Well, tripling the size of our object increases its mass by a factor of 2 (since the new object is simply made up of two of the smaller ones, plus some empty space in the middle), so the dm picks up a factor of 2. Thus the I for an object of length 3ℓ is 18 times the I for an object of length ℓ (where the axes pass through any two corresponding points).

With pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious (moments of inertia add), and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate • – – gives



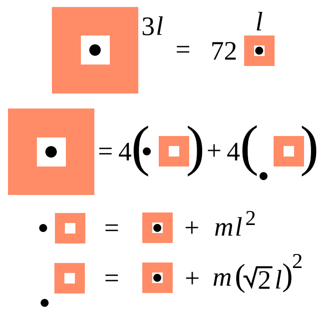
This is larger than the I for a uniform stick ($m\ell^2/12$), because the mass is generally further away from the center.

Remark: When we increase the length of our object by a factor of 3 here, the factor of 2 in the dm is between the factor of 1 relevant to a zero-dimensional object, and the factor of 3 relevant to a one-dimensional object. So in some sense our object has a dimension between 0 and 1. It is reasonable to define the dimension, d, of an object as the number for which r^d is the increase in 'volume' when the dimensions are increased by a factor of r. In our example, we have $3^d = 2$, so $d = \log_3 2 \approx 0.63$. ♣

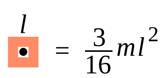
(b) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 3 and see what happens to I. In the integral $\int x^2 dm$, the

x's pick up a factor of 3, so this gives a factor of 9. But what happens to the dm? Tripling the size of our object increases its mass by a factor of 8 (since the new object is made up of eight of the smaller ones, plus an empty square in the middle), so the dm picks up a factor of 8. Thus the I for an object of side 3ℓ is 72 times the I for an object of side ℓ (where the axes pass through any two corresponding points).

Again, with pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious, and the third and fourth come from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third and fourth to eliminate  and  gives

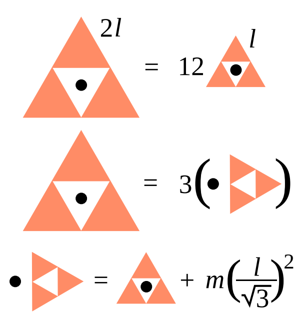


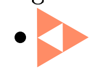
This is larger than the I for the uniform square in problem 4, because the mass is generally further away from the center.

Note: Increasing the size of our object by a factor of 3 increases the 'volume' by a factor of 8. So the dimension is given by $3^d = 8$; hence $d = \log_3 8 \approx 1.89$.

(c) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 2 and see what happens to I. In the integral $\int x^2 dm$, the x's pick up a factor of 2, so this gives a factor of 4. But what happens to the dm? Doubling the size of our object increases its mass by a factor of 3 (since the new object is simply made up of three of the smaller ones, plus an empty triangle in the middle), so the dm picks up a factor of 3. Thus the I for an object of side 2ℓ is 12 times the I for an object of side ℓ (where the axes pass through any two corresponding points).

Again, with pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious, and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

 $= \frac{1}{9} ml^2$

l

This is larger than the I for the uniform triangle in problem 4, because the mass is generally further away from the center.

Note: Increasing the size of our object by a factor of 2 increases the 'volume' by a factor of 3. So the dimension is given by $2^d = 3$; hence $d = \log_2 3 \approx 1.58$.

6. **Minimum I**

The shape should be a cylinder with the z-axis as its symmetry axis. This is fairly obvious, and a quick proof (by contradiction) is the following.

Assume the optimal blob is not a cylinder, and consider the surface of the blob. If the blob is not a cylinder, then there exist two points on the surface, P_1 and P_2, that are located at different distances, r_1 and r_2, from the z-axis. Assume $r_1 < r_2$ (see Fig. 7.35). Then moving a small piece of the blob from P_2 to P_1 will decrease the moment of inertia, $\int r^2 \rho \, dV$. Hence, the proposed blob was not the one with smallest I.

In order to avoid this contradiction, we must have all points on the surface be equidistant from the z-axis. The only blob with this property is a cylinder.

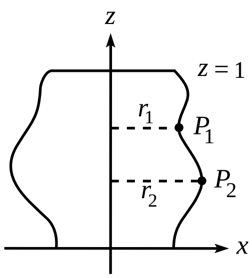


Figure 7.35

7. **Removing a support**

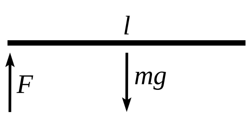


Figure 7.36

(a) **First Solution:** Let the desired force on the left support be F. Let the acceleration of the CM of the stick be a. Then (looking at torques around the CM, to obtain the second equation; see Fig. 7.36),

$$\begin{aligned} mg - F &= ma, \\ F\frac{\ell}{2} &= \frac{m\ell^2}{12}\alpha, \\ a &= \frac{\ell}{2}\alpha. \end{aligned} \tag{7.48}$$

Solving for F gives $F = mg/4$. (So the CM accelerates at $3g/4$, and the right end accelerates at $3g/2$.)

Second Solution: Looking at torques around the CM, we have

$$F\frac{\ell}{2} = \frac{m\ell^2}{12}\alpha. \tag{7.49}$$

Looking at torques around the fixed end, we have

$$mg\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha. \tag{7.50}$$

These two equations give $F = mg/4$.

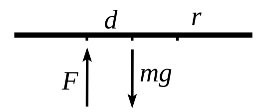


Figure 7.37

(b) **First Solution:** As in the first solution above, we have (see Fig. 7.37)

$$\begin{aligned} mg - F &= ma, \\ Fd &= (\eta m r^2)\alpha, \\ a &= d\alpha. \end{aligned} \quad (7.51)$$

Solving for F gives $F = mg(1 + d^2/\eta r^2)^{-1}$. For $d = r$ and $\eta = 1/3$, we get the answer in part (a).

Second Solution: As in the second solution above, looking at torques around the CM, we have

$$Fd = (\eta m r^2)\alpha. \quad (7.52)$$

Looking at torques around the fixed pivot, we have

$$mgd = (\eta m r^2 + md^2)\alpha. \quad (7.53)$$

These two equations give $F = mg(1 + d^2/\eta r^2)^{-1}$.

Some limits: If $d = r$, then: in the limit $\eta = 0$, $F = 0$; if $\eta = 1$, $F = mg/2$; and in the limit $\eta = \infty$, $F = mg$; these all make sense. In the limit $d = 0$, $F = mg$. And in the limit $d = \infty$, $F = 0$. (More precisely, we should be writing $d \ll \sqrt{\eta}\, r$ or $d \gg \sqrt{\eta}\, r$.)

8. **Oscillating ball**

Let the angle from the bottom of the cylinder be θ (see Fig. 7.38). Let F_f be the friction force. Then $F = ma$ gives

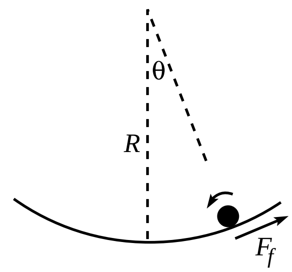


Figure 7.38

$$F_f - mg\sin\theta = ma. \quad (7.54)$$

Looking at torque and angular momentum around the CM, we have

$$-rF_f = \frac{2}{5} m r^2 \alpha. \quad (7.55)$$

Using $r\alpha = a$, this equation gives $F_f = -2ma/5$. Plugging this into eq. (7.54), and using $\sin\theta \approx \theta$, yields $mg\theta + 7ma/5 = 0$. Under the assumption $r \ll R$, we have $a \approx R\ddot{\theta}$, so we finally have

$$\ddot{\theta} + \left(\frac{5g}{7R}\right)\theta = 0. \quad (7.56)$$

This is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5g}{7R}}. \quad (7.57)$$

This answer is slightly smaller than the $\sqrt{g/R}$ answer if the ball were sliding. (The rolling ball effectively has a larger inertial mass, but the same gravitational mass.)

This problem can also be done using the contact point as the origin around which τ and L are calculated.

REMARK: If we get rid if the $r \ll R$ assumption, we leave it to you to show that $r\alpha = a$ still holds, but $a = R\ddot{\theta}$ changes to $a = (R - r)\ddot{\theta}$. Therefore, the exact result for the frequency is $\omega = \sqrt{5g/7(R-r)}$. This goes to infinity as r gets close to R. ♣

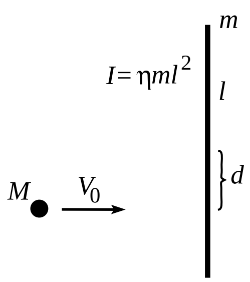


Figure 7.39

9. **Ball hitting stick**

Let V be the speed of the ball after the collision. Let v be the speed of the CM of the stick after the collision. Let ω be the angular speed of the stick after the collision. Conservation of momentum, angular momentum (around the initial center of the stick), and energy give (see Fig. 7.39)

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + \eta m\ell^2\omega, \\ MV_0^2 &= MV^2 + mv^2 + \eta m\ell^2\omega^2. \end{aligned} \tag{7.58}$$

We must solve these three equations for V, v, and ω. The first two equations quickly give $vd = \eta\ell^2\omega$. Solving for V in the first equation and plugging it into the third, and then eliminating ω through $vd = \eta\ell^2\omega$ gives

$$v = V_0\frac{2}{1+\frac{m}{M}+\frac{d^2}{\eta\ell^2}}, \qquad \text{and thus } \; v = V_0\frac{2\frac{d}{\eta\ell^2}}{1+\frac{m}{M}+\frac{d^2}{\eta\ell^2}}. \tag{7.59}$$

Knowing v, the first equation above gives V as

$$V = V_0\frac{1-\frac{m}{M}+\frac{d^2}{\eta\ell^2}}{1+\frac{m}{M}+\frac{d^2}{\eta\ell^2}}. \tag{7.60}$$

Another solution is of course $V = V_0$, $v = 0$, and $\omega = 0$. Nowhere in eqs. (7.58) does it say that the ball actually hits the stick.

The reader is encouraged to check various limits of these answers.

10. **A ball and stick theorem**

Let V be the speed of the ball after the collision. Let v be the speed of the CM of the stick after the collision. Let ω be the angular speed of the stick after the collision. Conservation of momentum, angular momentum (around the initial center of the stick), and energy give (see Fig. 7.39)

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + I\omega, \\ MV_0^2 &= MV^2 + mv^2 + I\omega^2. \end{aligned} \tag{7.61}$$

The speed of the contact point on the stick right after the collision is $v + \omega d$. So the desired relative speed is $(v + \omega d) - V$. We can solve the three above equations for V, v, and ω and obtain our answer (i.e., use the results of problem 9), but there's a slightly more appealing method.

The first two equations quickly give $mvd = I\omega$. The last equation may be written in the form (using $I\omega^2 = (I\omega)\omega = (mvd)\omega$)

$$M(V_0 - V)(V_0 + V) = mv(v + \omega d). \tag{7.62}$$

Dividing this by the first equation, written in the from $M(V_0 - V) = mv$, gives $V_0 + V = v + \omega d$, or

$$V_0 = (v + \omega d) - V, \tag{7.63}$$

as was to be shown.

11. **A triangle of circles**

(a) Let the normal force between the circles be N. Let the friction force from the ground be F_f (see Fig. 7.40). If we consider torques around the centers of the bottom balls, then the only force we have to worry about is F_f (since N, gravity, and the normal force from the ground point through the centers).

Let a_x be the initial horizontal acceleration of the right bottom circle (so $\alpha = a_x/R$ is its angular acceleration). Let a_y be the initial vertical acceleration of the top circle (downward taken to be positive). Then

$$\begin{aligned} N\cos 60^\circ - F_f &= Ma_x, \\ Mg - 2N\sin 60^\circ &= Ma_y, \\ F_f R &= (\eta MR^2)(a_x/R). \end{aligned} \tag{7.64}$$

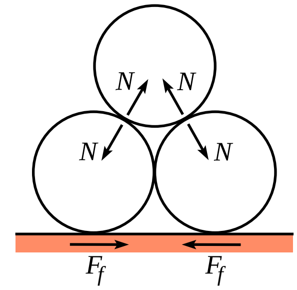


Figure 7.40

We have four unknowns, N, F_f, a_x, and a_y. So we need one more equation. Fortunately, a_x and a_y are related. The 'surface' of contact between the top and bottom circles lies at an angle of 30° with the horizontal. Therefore, if a bottom circle moves a distance d to the side, then the top circle moves a distance $d\tan 30^\circ$ downward. So

$$a_x = \sqrt{3}a_y. \tag{7.65}$$

We now have four equations and four unknowns. Solving for a_y, by your method of choice, gives

$$a_y = \frac{g}{7+6\eta}. \tag{7.66}$$

(b) Let the normal force between the circles be N. Let the friction between the circles be F_f (see Fig. 7.41). If we consider torques around the centers of the bottom balls, then the only force we have to worry about is F_f.

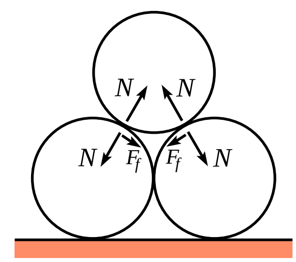


Figure 7.41

Let a_x be the initial horizontal acceleration of the right bottom circle. Let a_y be the initial vertical acceleration of the top circle (downward taken to be positive). From the same reasoning as in part (a), we have $a_x = \sqrt{3}a_y$. Let α be the angular acceleration of the right bottom circle (counterclockwise taken to be positive). Note that α is *not* equal to a_x/R, because the bottom circles slip. The four equations analogous to eqs. (7.64) and (7.65) are

$$\begin{aligned} N\cos 60^\circ - F_f\sin 60^\circ &= Ma_x, \\ Mg - 2N\sin 60^\circ - 2F_f\cos 60^\circ &= Ma_y, \\ F_f R &= (\eta MR^2)\alpha, \\ a_x &= \sqrt{3}a_y. \end{aligned} \tag{7.67}$$

We have five unknowns, N, F_f, a_x, a_y, and α. So we need one more equation. The tricky part is relating α to a_x. To do this, it is easiest to ignore the y motion of the top circle and imagine the bottom right circle to be rotating up and around the top circle, which is held fixed. If the bottom circle moves an infinitesimal distance d to the right, then its center moves a distance $d/\cos 30^\circ$ up and to the right. So the angle through which the bottom circle rotates is $d/(R\cos 30^\circ)$. Bringing back in the vertical motion of the balls does not change this result. Therefore,

$$\alpha = \frac{2}{\sqrt{3}}\frac{a_x}{R}. \tag{7.68}$$

We now have five equations and five unknowns. Solving for a_y, by your method of choice, gives

$$a_y = \frac{g}{7+8\eta}. \tag{7.69}$$

REMARK: If $\eta \neq 0$, this result is smaller than that in part (a). This is not all that intuitive. The basic reason is that the bottom circles in part (b) have to rotate a bit faster, so they take up more energy. Also, one can show that the N's are equal in (a) and (b), which is likewise not obvious. Since there's an extra force (from the friction) holding the top ball up in part (b), the acceleration is smaller. ♣

12. **Lots of sticks**

Consider the collision between two sticks. Let the speed of the end of the heavy one be V. Since this stick is essentially infinitely heavy, we may consider it to be an infinitely heavy ball, moving at speed V. (The rotational degree of freedom of the heavy stick is irrelevant, as far as the light stick is concerned.)

We will solve this problem by first finding the speed of the contact point on the light stick, and then finding the speed of the other end of the light stick.

- **Speed of contact point:**

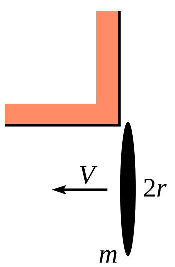


Figure 7.42

We can invoke the result of problem 10 to say that the point of contact on the light stick picks up a speed of $2V$. But let's prove this from scratch here in a different way: In the same spirit as the (easier) problem of the collision between two balls of greatly disparate masses, we will work things out in the rest frame of the infinitely heavy ball right before the collision. The situation then reduces to a stick of mass m, length $2r$, moment of inertia $\eta m r^2$, and speed V, approaching a fixed wall (see Fig. 7.42). To find the behavior of the stick after the collision, we will use (1) conservation of energy, and (2) conservation of angular momentum around the contact point.

Let u be the speed of the center of mass of the stick after the collision. Let ω be its angular velocity after the collision. Since the wall is infinitely heavy, it will acquire zero kinetic energy. So conservation of E gives

$$\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + \frac{1}{2}(\eta m r^2)\omega^2. \tag{7.70}$$

The initial angular momentum around the contact point is $L = mrV$, so conservation of L gives (breaking L after the collision up into the L of the CM plus the L relative to the CM)

$$mrV = mru + (\eta m r^2)\omega. \tag{7.71}$$

Solving eqs. (7.70) and (7.71) for u and $r\omega$ in terms of V gives

$$u = V\frac{1-\eta}{1+\eta}, \qquad \text{and} \qquad rw = V\frac{2}{1+\eta}. \tag{7.72}$$

(The other solution, $u = V$ and $r\omega = 0$ represents the case where the stick misses the wall.) The relative speed of the wall (i.e., the ball) and the point of contact on the light stick is

$$rw - u = V, \tag{7.73}$$

as was to be shown.

Going back to the lab frame (i.e., adding V onto this speed) shows that the point of contact on the light stick moves at speed $2V$.

- **Speed of other end:**

  Consider a stick struck at an end, with impulse $\mathcal{I}$. The speed of the CM is then $v_{\text{CM}} = \mathcal{I}/m$. The angular impulse is $\mathcal{I}r$, so $\mathcal{I}r = \eta m r^2 \omega$, and hence $r\omega = \mathcal{I}/m\eta = v_{\text{CM}}/\eta$.

  The speed of the struck end is $v_{\text{str}} = r\omega + v_{\text{CM}}$. The speed of the other end (taking positive to be in the reverse direction) is $v_{\text{oth}} = r\omega - v_{\text{CM}}$. The ratio of these is

$$\frac{v_{\text{oth}}}{v_{\text{str}}} = \frac{v_{\text{CM}}/\eta - v_{\text{CM}}}{v_{\text{CM}}/\eta + v_{\text{CM}}} = \frac{1-\eta}{1+\eta}. \tag{7.74}$$

  In the problem at hand, we have $v_{\text{str}} = 2V$. Therefore,

$$v_{\text{oth}} = V\frac{2(1-\eta)}{1+\eta}. \tag{7.75}$$

The same analysis works in all the other collisions. Therefore, the bottom ends of the sticks move with speeds that form a geometric progression with ratio $2(1-\eta)/(1+\eta)$. If this ratio is less than 1 (i.e., $\eta > 1/3$), then the speeds go to zero, as $n \to \infty$. If it is greater than 1 (i.e., $\eta < 1/3$), then the speeds go to infinity, as $n \to \infty$. If it equals 1 (i.e., $\eta = 1/3$), then the speeds are independent of n, as $n \to \infty$. Therefore,

$$\eta = \frac{1}{3} \tag{7.76}$$

is the desired answer. A uniform stick has $\eta = 1/3$ (usually written in the form $I = m\ell^2/12$, where $\ell = 2r$).

13. **Falling stick**

(a) It is easiest to calculate τ and L relative to the pivot point. The torque is due to gravity, which effectively acts on the CM. It has magnitude mgb.

The moment of inertia of the stick around a horizontal axis through the pivot (and perpendicular to the massless stick) is simply mb^2. So when the stick starts to fall, $\tau = dL/dt$ gives $mgb = (mb^2)\alpha$. Therefore, the initial acceleration of the CM, $b\alpha$, is

$$b\alpha = g, \tag{7.77}$$

independent of ℓ and b.

This makes sense. This stick initially falls straight down, and the pivot provides no force because it doesn't know right away that the stick is moving.

(b) The only change from part (a) is the moment of inertia of the stick around a horizontal axis through the pivot (and perpendicular to the massless stick). From the parallel axis theorem, this moment is $mb^2 + m\ell^2/12$. So when the stick starts to fall, $\tau = dL/dt$ gives $mgb = (mb^2 + m\ell^2/12)\alpha$. Therefore, the initial acceleration of the CM, $b\alpha$, is

$$b\alpha = \frac{g}{1 + \frac{\ell^2}{12b^2}}. \tag{7.78}$$

As $\ell \to 0$, this goes to g, as it should. As $\ell \to \infty$, it goes to 0, as it should (a tiny movement in the CM corresponds to a very large movement in the points far out along the stick).

14. **Falling Chimney**

Let the height of the chimney be ℓ. Let the width be $2r$. The moment of inertia around the pivot point is $m\ell^2/3$ (if we ignore the width). Let the angle with the vertical be θ. Then the torque (around the pivot point) due to gravity is $\tau = mg(\ell/2)\sin\theta$. So $\tau = dL/dt$ gives $mg(\ell/2)\sin\theta = (1/3)m\ell^2\ddot{\theta}$, or

$$\ddot{\theta} = \frac{3g\sin\theta}{2\ell}. \tag{7.79}$$

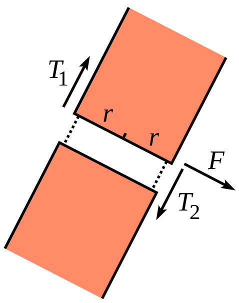


Figure 7.43

Consider the chimney to consist of a chimney of height a, with another one of height $\ell - a$ placed on top of it. We will find the tensions in the strings connecting these two 'sub-chimneys'; then we will maximize one of the tensions as a function of a.

The forces on the top piece are gravity and the forces on each end of the bottom board. Let us break these latter forces up into transverse and longitudinal forces along the chimney. Let T_1 and T_2 be the two longitudinal components, and let F be the sum of the transverse components. These are shown in Fig. 7.43. We have picked the positive directions for T_1 and T_2 such that positive T_1 corresponds to a normal force, and positive T_2 corresponds to a tension in the string. This is the case we will be concerned with. (If, for example, T_1 happens to be negative, then it simply corresponds to a tension instead of a normal force.) It turns out that if $r \ll \ell$, then $T_2 \gg F$ (as we will see below), so the tension in the right string is essentially equal to T_2. We will therefore be concerned with maximizing T_2.

In writing down the force and torque equations for the top piece, we have three equations ($F_x = ma_x$, $F_y = ma_y$, and $\tau = dL/dt$ around its center-of-mass), and three unknowns (F, T_1, and T_2). Using the fact that the top piece has length $(\ell - a)$, its CM travels in a circle of radius $(\ell + a)/2$, and its mass is $m(\ell - a)/\ell$, our three equations are, respectively,

$$\begin{aligned} (T_1 - T_2)\sin\theta + F\cos\theta &= \frac{m(\ell - a)}{\ell}\left(\frac{\ell + a}{2}\ddot{\theta}\cos\theta\right), \\ (T_1 - T_2)\cos\theta - F\sin\theta - \frac{mg(\ell - a)}{\ell} &= -\frac{m(\ell - a)}{\ell}\left(\frac{\ell + a}{2}\ddot{\theta}\sin\theta\right), \\ (T_1 + T_2)r - F\frac{\ell - a}{2} &= \frac{m(\ell - a)}{\ell}\left(\frac{(\ell - a)^2}{12}\ddot{\theta}\right). \end{aligned} \tag{7.80}$$

We can solve for F by multiplying the first equation by $\cos\theta$, the second by $\sin\theta$, and subtracting. Using (7.79) to eliminate $\ddot{\theta}$ gives

$$F = \frac{mg\sin\theta}{4}(-1 + 4f - 3f^2), \tag{7.81}$$

where $f \equiv a/\ell$ is the fraction of the way along the chimney.

We may now solve for T_2. Multiplying the second of eqs. (7.80) by r and subtracting from the third gives (to leading order in the large number ℓ/r)[9]

$$T_2 \approx F\frac{\ell - a}{4r} + \frac{m(\ell - a)}{\ell}\frac{(\ell - a)^2}{24r}\ddot{\theta}. \tag{7.82}$$

[9] This result is simply the third equation with T_1 set equal to T_2. Basically, T_1 and T_2 are both very large and are essentially equal; the difference between them is of order 1.

Using eqs. (7.79) and (7.81), this may be written as

$$T_2 \approx \frac{mg\ell \sin\theta}{8r} f(1-f)^2. \tag{7.83}$$

As stated above, this is much greater than F (since $\ell/r \gg 1$), so the tension in the right string is essentially equal to T_2. Taking the derivative with respect to f, we see that T_2 is maximum at

$$f \equiv \frac{a}{\ell} = \frac{1}{3}. \tag{7.84}$$

So our chimney is most likely to break at a point one-third of the way up (assuming that the width is much less than the height).

15. **Zero torque from internal forces**

Let $\mathbf{F}_{ij}^{\text{int}}$ be the force that the ith particle feels due to the jth particle (see Fig. 7.44. Then

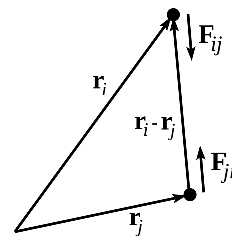


Figure 7.44

$$\mathbf{F}_i^{\text{int}} = \sum_j \mathbf{F}_{ij}^{\text{int}}, \tag{7.85}$$

and Newton's third law says that

$$\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}. \tag{7.86}$$

Therefore,

$$\boldsymbol{\tau}^{\text{int}} \equiv \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.87}$$

But if we change the indices (which were labeled arbitrarily), we have

$$\boldsymbol{\tau}^{\text{int}} = \sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} = -\sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.88}$$

Adding the two previous equations gives

$$2\boldsymbol{\tau}^{\text{int}} = \sum_i \sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.89}$$

But our central-force assumption says that $\mathbf{F}_{ij}^{\text{int}}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. Therefore, each cross-product in the sum is zero.

16. **Lengthening the pendulum** *

Consider the angular momentum, $\mathbf{L}$, around the support point, P. The forces on the mass are the tension in the string and gravity. The former provides no torque around P, and the latter provides no torque in the z-direction. Therefore, L_z is constant.

Let ω_ℓ be the frequency of the circular motion, when the string has length ℓ. Then

$$mr^2\omega_\ell = L_z \tag{7.90}$$

is constant.

The frequency ω_ℓ is obtained by using $F = ma$ for the circular motion. The tension in the string is $mg/\cos\theta$, so the horizontal radial force is $mg\tan\theta$. Therefore,

$$mg\tan\theta = mr\omega_\ell^2 = m(\ell\sin\theta)\omega_\ell^2 \qquad \Longrightarrow \qquad \omega_\ell = \sqrt{\frac{g}{\ell\cos\theta}}. \tag{7.91}$$

plugging this into eq. (7.90) gives

$$mr^2\sqrt{\frac{g}{\ell\cos\theta}} = mr^2\sqrt{\frac{g}{h}} = L_z. \tag{7.92}$$

(a) For $\theta \approx 0$, we have $h \approx \ell$, so eq. (7.92) gives $r^2/\sqrt{\ell} \approx C$. Therefore,

$$r \propto \ell^{1/4}. \tag{7.93}$$

So r grows very slowly with ℓ.

(b) For $\theta \approx \pi/2$, we have $r \approx \ell$, so eq. (7.92) gives $\ell^2/\sqrt{h} \approx C$. Therefore,

$$h \propto \ell^4. \tag{7.94}$$

So h grows very quickly with ℓ.

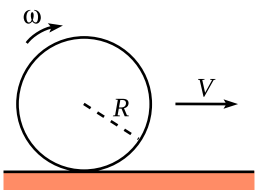


Figure 7.45

17. **Sliding to rolling**

(a) Let the ball travel to the right. Define all linear quantities to be positive to the right, and all angular quantities to be positive clockwise, as shown in Fig. 7.45. (Then, for example, the friction force F_f is negative.) The friction force slows down the translational motion and speeds up the rotational motion, according to (looking at torque around the CM)

$$\begin{aligned} F_f &= ma, \\ -F_f R &= I\alpha. \end{aligned} \tag{7.95}$$

Eliminating F_f, and using $I = \eta m R^2$, gives $a = -\eta R\alpha$. Integrating this over time, up to the time when the ball stops slipping, gives

$$\Delta V = -\eta R \Delta\omega. \tag{7.96}$$

(This is the same statement as the impulse equation, eq. (7.31).) Using $\Delta V = V_f - V_0$, and $\Delta\omega = \omega_f - \omega_0 = \omega_f$, and also $\omega_f = V_f/R$ (the non-slipping condition), we find

$$V_f = \frac{V_0}{1+\eta}, \tag{7.97}$$

independent of how F_f depends on position. (For that matter, F_f could even depend on time or speed. The relation $a = -\eta R\alpha$ would still be true at all times, and hence also eq. (7.96).)

Remark: You could also calculate τ and L relative to the instantaneous point of contact on the ground (which is a fixed point). There is zero torque relative to this point. The motion around this point is not a simple rotation, so we have to add the L of the CM plus the L relative to the CM. $\tau = dL/dt$ gives $0 = (d/dt)(mRv + \eta m R^2\omega)$. Hence, $a = -\eta R\alpha$.

Note that it is *not* valid to calculate τ and L relative to the instantaneous point of contact *on the ball*. The ball is slowing down, so there is a horizontal component to the acceleration, and hence the third condition in section 7.3.3 does not hold. ♣

The loss in kinetic energy is given by (using eq. (7.97), and also the relation $\omega_f = V_f/R$)

$$\begin{aligned} \Delta KE &= \frac{1}{2}mV_0^2 - \left(\frac{1}{2}mV_f^2 + \frac{1}{2}I\omega_f^2\right) \\ &= \frac{1}{2}mV_0^2\left(1 - \frac{1}{(1+\eta)^2} - \frac{\eta}{(1+\eta)^2}\right) \\ &= \frac{1}{2}mV_0^2\left(\frac{\eta}{1+\eta}\right). \end{aligned} \tag{7.98}$$

For $\eta \to 0$, no energy is lost, which makes sense. And for $\eta \to \infty$, all the energy is lost, which also makes sense (this case is essentially like a sliding block which can't rotate).

(b) Let's find t. The friction force is $F_f = -\mu mg$. So $F = ma$ gives $-\mu g = a$ (so a is constant). Therefore, $\Delta V = at = -\mu gt$. But eq. (7.97) says that $\Delta V \equiv V_f - V_0 = -V_0\eta/(1+\eta)$. So we find

$$t = \frac{\eta}{\mu(1+\eta)}\frac{V_0}{g}. \tag{7.99}$$

For $\eta \to 0$, we have $t \to 0$, which makes sense. And for $\eta \to \infty$, we have $t \to V_0/(\mu g)$ which is exactly the time a sliding block would take to stop.

Now let's find d. We have $d = V_0 t + (1/2)at^2$. Using $a = -\mu g$, and plugging in t from above gives

$$d = \frac{V_0^2}{g}\frac{\eta(2+\eta)}{2\mu(1+\eta)^2}. \tag{7.100}$$

The two extreme cases for η check here.

To calculate the work done by friction, one might be tempted to take the product $F_f d$. But the result doesn't look much like the loss in kinetic energy calculated in eq. (7.98). What's wrong with this? The error is that the friction force does not act over a distance d. To find the distance over which F_f acts, we must find how far the surface of the ball moves relative to the ground.

The relative speed of the point of contact and the ground is $V_{\text{rel}} = V(t) - R\omega(t) = (V_0 + at) - R\alpha t$. Using $a = -\eta R\alpha$ and $a = -\mu g$, this becomes

$$V_{\text{rel}} = V_0 - \frac{1+\eta}{\eta}\mu g t. \tag{7.101}$$

Integrating this from $t = 0$ to the t given in eq. (7.99) yields

$$d_{\text{rel}} = \int V_{\text{rel}} dt = \frac{V_0^2\eta}{2\mu g(1+\eta)}. \tag{7.102}$$

The work done by friction is $F_f d_{\text{rel}} = \mu m g d_{\text{rel}}$, which does indeed give the ΔKE in eq. (7.98).

18. **The superball**

The y-motion of the ball is irrelevant in this problem, because the y-velocity simply reverses direction, and the vertical impulse from the floor provides no torque around the CM of the ball.

With the positive directions for x and ω as stated in the problem, eq. (7.31) may be used to show that the horizontal impulse from the floor changes v_x and ω according to

$$I(\omega' - \omega) = -Rm(v_x' - v_x). \tag{7.103}$$

The conservation-of-energy statement is

$$\frac{1}{2}m{v_x'}^2 + \frac{1}{2}I\omega'^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2. \tag{7.104}$$

Given v_x and ω, eqs. (7.103) and (7.104) are two equations in the two unknowns v_x' and ω'. They can be solved in a messy way using the quadratic formula, but it is much easier to use the standard trick of rewriting eq. (7.104) as

$$I(\omega'^2 - \omega^2) = -m({v_x'}^2 - v_x^2), \tag{7.105}$$

and then dividing this by eq. (7.103) to obtain

$$R(\omega' + \omega) = (v'_x + v_x). \tag{7.106}$$

Eqs. (7.106) and (7.103) are now two linear equations in the two unknowns v'_x and ω'. Using $I = (2/5)mR^2$ for a solid sphere, you can easily solve the equations to obtain the desired result, eq. (7.32).

REMARK: The other solution to eqs. (7.103) and (7.104) is of course $v'_x = v_x$ and $\omega' = \omega$. This corresponds to the ball bouncing off a frictionless floor (or even just passing through the floor). Eq. (7.103) is true for any ball, but the conservation-of-energy statement in eq. (7.104) is only true for two special cases. One is the case of a frictionless floor, where there is "maximal" slipping at the point of contact. The other is the case of zero slipping, which the case with the superball. If there is any intermediate amount of slipping, then energy is not conserved, because the friction force does work and generates heat. (Work is force times distance, and in the first special case, the force is zero; while in the second special case, the distance is zero). Therefore, in addition to being made of a very bouncy material, a superball must also have a surface that won't slip while in contact with the floor.

Note that eq. (7.106) may easily be used to show that the relative velocity of the point of contact and the ground exactly reverses direction during the bounce. ♣

19. **Many bounces**

Eq. (7.32) gives the result after one bounce, so the result after two bounces is

$$\begin{aligned}
\begin{pmatrix} v''_x \\ R\omega'' \end{pmatrix} &= \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix} \begin{pmatrix} v'_x \\ R\omega' \end{pmatrix} \\
&= \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix}^2 \begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\
&= \begin{pmatrix} v_x \\ R\omega \end{pmatrix}.
\end{aligned} \tag{7.107}$$

The square of the matrix turns out to be the identity. Therefore, after two bounces, both v_x and ω return to their original values. The ball then repeats the motion of the previous two bounces (and so on, after every two bounces). The only difference between successive pairs of bounces is that the ball may shift horizontally. You are strongly encouraged to experimentally verify this strange periodic behavior.

20. **Rolling over a bump**

We will use the fact that the angular momentum of the ball with respect to the corner of the point (call it point P) is unchanged by the collision. This is true because any forces exerted at point P provide zero torque around P.[10] This fact will allow us to find the energy of the ball right after the collision, which we will then require to be greater than Mgh.

Breaking L into the contribution relative to the CM, plus the contribution from the ball treated like a point mass located at its CM (eq. (7.9)), gives an initial angular momentum equal to $L = (2/5)MR^2\omega_0 + MV_0(R - h)$, where ω_0 is the initial rolling

[10]The torque from gravity will be relevant once the ball rises up off the ground. But during the (instantaneous) collision, L will not change.

angular speed. But the non-slipping condition requires that $V_0 = R\omega_0$. Hence, L may be written as

$$L = \frac{2}{5}MRV_0 + MV_0(R-h) = MV_0\left(\frac{7R}{5} - h\right). \qquad (7.108)$$

Let ω' be the angular speed of the ball around point P immediately after the collision. The parallel-axis theorem says that the ball's moment of inertia around P is equal to $(2/5)MR^2 + MR^2 = (7/5)MR^2$. Conservation of L (around point P) during the collision then gives

$$MV_0\left(\frac{7R}{5} - h\right) = \frac{7}{5}MR^2\omega', \qquad (7.109)$$

which gives ω'. The energy of the ball right after the collision is therefore

$$E = \frac{1}{2}\left(\frac{7}{5}MR^2\right)\omega'^2 = \frac{1}{2}\left(\frac{7}{5}MR^2\right)\left(\frac{MV_0(7R/5-h)}{(7/5)MR^2}\right)^2 = \frac{MV_0^2(7R/5-h)^2}{(14/5)R^2}. \qquad (7.110)$$

The ball will climb up over the step if $E \geq Mgh$, which gives

$$V_0 \geq \frac{R\sqrt{14gh/5}}{7R/5-h}. \qquad (7.111)$$

REMARKS: It is indeed possible for the ball to rise up over the step, even if $h > R$ (as long as the ball sticks to the corner, without slipping). But note that $V_0 \to \infty$ as $h \to 7R/5$. For $h \geq 7R/5$, it is impossible for the ball to make it up over the step. (The ball will actually get pushed down into the ground, instead of rising up, if $h > 7R/5$.)

For an object with a general moment of inertia $I = \eta MR^2$ (so $\eta = 2/5$ in our problem), you can easily show that the minimum initial speed is

$$V_0 \geq \frac{R\sqrt{2(1+\eta)gh}}{(1+\eta)R-h}. \qquad (7.112)$$

This decreases as η increases. It is smallest when the "ball" is a wheel with all the mass on its rim (so that $\eta = 1$), in which case it is possible for the wheel to climb over the step even if h approaches $2R$. ♣