Chapter 5

The Lagrangian Method

Consider the setup with a mass on the end of a spring. We can, of course, use $F = ma$ to write $m\ddot{x} = -kx$. The solutions to this are sinusoidal functions, as we well know. We can, however, solve this problem in another way which doesn't explicitly use $F = ma$. In many (in fact, probably most) physical situations, this new method is far superior to using $F = ma$. You will soon discover this for yourself when you tackle the problems for this chapter.

We will introduce the method in a slightly unconventional manner. We will state the procedure by pulling it completely out of the blue, at which point you will most likely feel some combination of suspicion, bewilderment, and amazement. We will then give the method proper justification.

5.1 The Euler-Lagrange equations

Here is the procedure. Just take it on faith for now. Form the following seemingly silly combination of the kinetic and potential energies (T and V, respectively),

$$\boxed{L \equiv T - V}. \tag{5.1}$$

This is called the *Lagrangian*. Yes, there is a minus sign in the definition (a plus sign would just give the total energy). In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.2}$$

Now write

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}. \tag{5.3}$$

(Don't worry, we'll show you in a little while where this comes from.) This is called the *Euler-Lagrange (E-L) equation.* For the problem at hand, we have $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial x = -kx$, so we obtain

$$m\ddot{x} = -kx, \tag{5.4}$$

exactly the result obtained using $F = ma$. An equation such as eq. (5.4), which is derived from eq. (5.3), is called an *equation of motion.*[1]

If the problem involves more than one coordinate, as most problems do, you simply have to apply eq. (5.3) to each coordinate. You will obtain as many equations as there are coordinates.

At this point, you may be thinking, "That was a nice little trick, but we just got lucky here; the procedure won't work for a more general problem." Well, let's see. How about if we consider the more general problem of a particle moving in an arbitrary potential, $V(x)$. (We'll stick to one dimension for now). Then the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \tag{5.5}$$

The Euler-Lagrange equation, eq. (5.3) gives

$$m\ddot{x} = -\frac{dV}{dx}. \tag{5.6}$$

But $-dV/dx$ is simply the force on the particle. So we see that eqs. (5.1) and (5.3) together say exactly the same thing as $F = ma$ (when using a cartesian coordinate in one dimension.)

Note that shifting the potential by a given constant clearly has no effect on the equation of motion, since it involves only derivatives of V. This is, of course, the same as saying that only the differences in energy are relevant, and not the actual values.

In the three-dimensional case, where the potential takes the form $V(x, y, z)$, it immediately follows that the three E-L equations may be combined into the vector statement, $-\nabla V = m\ddot{\mathbf{x}}$. That is, $\mathbf{F} = m\mathbf{a}$.

Let's do one more example to convince you that there's really something non-trivial going on here.

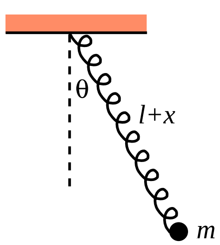

Figure 5.1

Example (Spring pendulum): Consider a pendulum made out of a spring with a mass m on the end (see Fig. 5.1). The spring is arranged to lie in a straight line. (We can do this by, say, wrapping the spring around a rigid massless rod.) The equilibrium length of the spring is ℓ. Let the spring have length $\ell + x(t)$, and let its angle with the vertical be $\theta(t)$. Find the equations of motions for x and θ.

Solution: The kinetic energy may be broken up into its radial and tangential parts, so we have

$$T = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big). \tag{5.7}$$

The potential energy comes from both gravity and the spring, so

$$V(x, \theta) = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2. \tag{5.8}$$

[1] The term "equation of motion" is slightly ambiguous. It is understood to refer to the second-order differential equation satisfied by x, and *not* the actual equation for x as a function of t, namely $x(t) = A\cos(\omega t + \phi)$ (which is obtained by integrating the equation of motion twice).

Therefore, the Lagrangian is

$$L = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big) + mg(\ell + x)\cos\theta - \frac{1}{2}kx^2. \tag{5.9}$$

There are two variables here, x and θ. The nice thing about the Lagrangian method is that you can simply use eq. (5.3) twice, once with x and once with θ. Hence, the two Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad m\ddot{x} = m(\ell + x)\dot{\theta}^2 + mg\cos\theta - kx, \tag{5.10}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad &\Longrightarrow \qquad \frac{d}{dt}\Big(m(\ell + x)^2\dot{\theta}\Big) = -mg(\ell + x)\sin\theta \\ &\Longrightarrow \qquad m(\ell + x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta. \end{aligned} \tag{5.11}$$

Eq. (5.10) is simply the radial $F = ma$ equation (complete with the centripetal acceleration, $(\ell + x)\dot{\theta}^2$). The first line of eq. (5.11) is the statement that torque equals the rate of change of angular momentum (topics of Chapter 7).[2]

After writing down the *E-L* equations, it is always best to double-check them by trying to identify them as $F = ma$ or $\tau = dL/dt$ equations. Sometimes, however, this identification is not obvious. For the times where everything is clear (that is, when you look at the *E-L* equations and say, "Oh . . . of course!"), it is usually clear only *after* you've derived the E-L equations. The Lagrangian method is generally the safer method to use.

The present example should convince you of the great utility of the Lagrangian method. Even if you've never heard of the terms "torque", "centripetal", "centrifugal", or "Coriolis", you can still get the correct answer by simply writing down the kinetic and potential energies, and then taking some derivatives.

At this point, it seems to be personal preference, and all academic, whether you use the Lagrangian method or the $F = ma$ method. The two methods produce the same equations. However, in problems involving more than one variable, it usually turns out to be *much* easier to write down T and V, as opposed to writing down all the forces. This is because T and V are nice and simple scalars. The forces, on the other hand, are vectors, and it's easy to get confused if they point in various directions. The Lagrangian method has the advantage that once you've written down $L = T - V$, you don't have to think anymore. All that remains to be done is to blindly take some derivatives. (Of course, you have to eventually solve the resulting equations of motion, but you have to do that when using the $F = ma$ method, too.)

But ease-of-computation aside, is there any fundamental difference between the two methods? Is there any deep reasoning behind eq. (5.3)? Indeed, there is . . .

[2] Alternatively, if you want to work in a rotating frame, then eq. (5.10) is the radial $F = ma$ equation, complete with the centrifugal force, $m(\ell + x)\dot{\theta}^2$. And the second line of eq. (5.11) is the tangential $F = ma$ equation, complete with the Coriolis force, $-2m\dot{x}\dot{\theta}$. But never mind about this now; we'll deal with rotating frames in Chapter 9.

5.2 The principle of stationary action

Consider the quantity,

$$S \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt. \tag{5.12}$$

S is called the *action*. It is a number with the dimensions of (Energy) × (Time). S depends on L, and L in turn depends on the function $x(t)$ via eq. (5.1).[3] Given any function $x(t)$, we can produce the number S.

S is called a *functional*, and is sometimes denoted $S[x(t)]$. It depends on the entire function $x(t)$, and not on just one input number, as a regular function $f(t)$ does. S can be thought of as a function of an infinite number of values, namely all the $x(t)$ for t ranging from t_1 to t_2. (If you don't like infinities, you can imagine breaking up the time interval into, say, a million pieces, and then replacing the integral by a discrete sum.)

Let us now pose the following question: Consider a function $x(t)$, for $t_1 \le t \le t_2$, which has its endpoints fixed (that is, $x(t_1) = x_1$ and $x(t_2) = x_2$), but is otherwise arbitrary. What function $x(t)$ yields a stationary point of S? (A stationary point is a local minimum, maximum, or saddle point.[4])

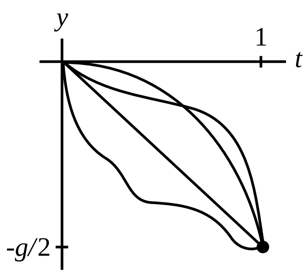

Figure 5.2

For example, consider a ball dropped from rest, and look at $y(t)$ for $0 \le t \le 1$. Assume that we know that $y(0) = 0$ and $y(1) = -g/2$. A number of possible functions are shown in Fig. 5.2, and each of these can (in theory) be plugged into eqs. (5.1) and (5.12) to generate S. Which one yields a stationary value of S?

Theorem 5.1 *If the function $x_0(t)$ yields a stationary value (that is, a local minimum, maximum, or saddle point) for S, then*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}. \tag{5.13}$$

(It is understood that we are considering the class of functions whose endpoints are fixed. That is, $x(t_1) = x_1$ and $x(t_2) = x_2$.)

Proof: We will use the fact that if a certain function $x_0(t)$ yields a stationary value of S, then another function very close to $x_0(t)$ (with the same endpoint values) will yield essentially the same S, up to first order in any deviations. (This is actually the definition of a stationary value.) The analogy with regular functions is that if $f(b)$ is a stationary value of f, then $f(b+\delta)$ differs from $f(b)$ only at second order in the small quantity δ. (This is true because $f'(b) = 0$, so there is no first-order term in the Taylor series.)

Assume that the function $x_0(t)$ yields a stationary value of S, and consider the function

$$x_a(t) \equiv x_0(t) + a\beta(t), \tag{5.14}$$

[3]In some situations, the kinetic and potential energies in $L \equiv T - V$ may explicitly depend on time, so we have included the "t" in eq. (5.12).

[4]A saddle point is a point where there are no first-order changes in S, and where some of the second-order changes are positive and some are negative (like the middle of a saddle, of course).

where $\beta(t)$ satisfies $\beta(t_1) = \beta(t_2) = 0$ (to keep the endpoints of the function the same), but is otherwise arbitrary.

The action $S[x_a(t)]$ is a function of a (the "t" is integrated out, so S is just a number, and it depends on a), and we demand that there be no change in S at first order in a. How does S depend on a? Using the chain rule, we have

$$\begin{aligned}\frac{d}{da}S[x_a(t)] &= \frac{d}{da}\int_{t_1}^{t_2} L\,dt \\ &= \int_{t_1}^{t_2} \frac{dL}{da}\,dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial a}\right) dt. \end{aligned} \tag{5.15}$$

From eq. (5.14), we have

$$\frac{\partial x_a}{\partial a} = \beta, \qquad \text{and} \qquad \frac{\partial \dot{x}_a}{\partial a} = \dot{\beta}, \tag{5.16}$$

so eq. (5.15) becomes[5]

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\right) dt. \tag{5.17}$$

Now comes the one sneaky part of the proof. (You will see this trick many times in your physics career.) Integrate the second term by parts. This yields

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}\right)\beta\,dt + \frac{\partial L}{\partial \dot{x}_a}\beta\Big|_{t_1}^{t_2}. \tag{5.18}$$

But $\beta(t_1) = \beta(t_2) = 0$, so the boundary term vanishes. We now use the fact that $(d/da)S[x_a(t)]$ must be zero for *any* function $\beta(t)$ (assuming that $x_0(t)$ yields a stationary value). The only way this can be true is if the quantity in parentheses above (evaluated at $a = 0$) equals zero, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}. \qquad \blacksquare \tag{5.19}$$

The theorem implies the following. If we accept that $F = ma$ is equivalent to the *E-L* equation, eq. (5.13), for any choice of coordinates (we showed this for cartesian coordinates in Section 5.1, and we'll prove it for any coordinate system in Section 5.4), then we may replace Newton's laws by the following principle.

- **The Principle of Stationary-Action:**

  The path of a particle is the one that yields a stationary value of the action.

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[5] Note that nowhere do we assume that x_a and $\dot{x}_a$ are independent variables. The partial derivatives in eq. (5.16) are very much related, in that one is the derivative of the other. The use of the chain rule in eq. (5.15) is still perfectly valid.

This principle is equivalent to Newton's laws because Theorem 5.1 shows that if (and only if, as you can easily show) we have a stationary value of S, then the E-L equations hold. And the E-L equations are (as we'll show in Section 5.4 for all coordinates) equivalent to $F = ma$. So "stationary-action" is equivalent to $F = ma$.

Consider the example of a ball dropped from rest, mentioned above. You are encouraged to verify explicitly that the path obtained from eq. (5.19) (or equivalently $F = ma$), namely $y(t) = -gt^2/2$, yields an action that is smaller (the stationary point happens to be a minimum here) than the action obtained from, say, the path $y(t) = -gt/2$ (which also satisfies the endpoint conditions). Any other such path you choose will also yield an action larger than the action for $y(t) = -gt^2/2$.

The E-L equation, eq. (5.3), therefore doesn't just come out of the blue. It is a necessary consequence of requiring the action to have a stationary value.

Remarks:

1. Admittedly, Theorem 5.1 simply shifts the burden of proof. We are now left with the task of justifying why we should want the action to have a stationary value. The good news is that there is a very solid reason for wanting this. The bad news is that the reason involves quantum mechanics, so we won't be able to discuss it properly here. Suffice it to say that a particle actually takes all possible paths in going from one place to another, and each path is associated with the complex number $e^{iS/\hbar}$ (where $\hbar = 1.05 \cdot 10^{-34}$ Js is *Planck's constant*). These "phases" are complex numbers with absolute value equal to 1. It turns out that the phases from all possible paths must be added up to give the "amplitude" of going from one point to another. The absolute value of the amplitude must then be squared to obtain the probability.[6]

   The basic point, then, is that at a non-stationary value of S the phases differ from one another (greatly, since $\hbar$ is very small), which effectively leads to the addition of many random vectors in the complex plane. These end up canceling each other, yielding a sum of essentially zero. There is therefore no contribution to the overall amplitude, from non-stationary values of S. So we do not observe the paths associated with these S's. At a stationary value of S, however, all the phases take on essentially the same value, thereby adding constructively instead of destructively. There is therefore a non-zero probability of the particle taking the path that yields a stationary value of S. So this is the path we observe.

2. Admittedly, again, the preceding remark simply shifts the burden of proof one step further. We must now justify why these phases $e^{iS/\hbar}$ should exist, and why the Lagrangian that appears in them should equal $T - V$. But here's where we're going to stop.

3. Our principle of stationary action is often referred to as the principle of "least" action. This is misleading. True, most of the time the stationary value turns out to be a minimum value, but it need not be, as we can see in the following example.

   Consider a harmonic oscillator. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.20}$$

[6]This is one of those remarks that won't please anyone, since it will be incomprehensible if you haven't come across this topic before, or trivial if you have. My apologies. But this and the following remarks are by no means necessary for an understanding of the material in this chapter. If you're interested in reading more about these quantum mechanics issues, you should take a look at Richard Feynman's book, *QED*. (Feynman was, after all, the one who thought of this idea.)

Let $x_0(t)$ be a function which yields a stationary value of the action. Then we know that $x_0(t)$ satisfies the E-L equation, $m\ddot{x}_0 = -kx_0$.

Consider a slight variation on this path, $x_0(t) + \xi(t)$, where $\xi(t)$ satisfies $\xi(t_1) = \xi(t_2) = 0$. With this new function, the action becomes

$$S_\xi = \int_{t_1}^{t_2} \left(\frac{m}{2}\left(\dot{x}_0^2 + 2\dot{x}_0\dot{\xi} + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + 2x_0\xi + \xi^2\right)\right) dt. \tag{5.21}$$

The two cross-terms add up to zero, because after integrating the $\dot{x}_0\dot{\xi}$ term by parts, their sum is

$$m\dot{x}_0\xi\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (m\ddot{x}_0 + kx_0)\xi\, dt. \tag{5.22}$$

The first term is zero, due to the boundary conditions on $\xi(t)$. The second term is zero, due to the E-L equation. (We've basically just reproduced the proof of Theorem 5.1 for the special case of the harmonic oscillator here.)

The terms involving only x_0 give the stationary value of the action (call it S_0). To determine whether S_0 is a minimum, maximum, or saddle point, we must look at the difference,

$$\Delta S \equiv S_\xi - S_0 = \frac{1}{2}\int_{t_1}^{t_2} (m\dot{\xi}^2 - k\xi^2)\, dt. \tag{5.23}$$

It is always possible to find a function ξ that makes ΔS positive. (Simply choose ξ to be small, but make it wiggle very fast, so that $\dot{\xi}$ is large). Therefore, it is *never* the case that S_0 is a maximum. (This reasoning works for any potential, not just a harmonic oscillator.)

You might be tempted to use the same reasoning to say that it is also always possible to find a function ξ that makes ΔS negative, by making ξ large and $\dot{\xi}$ small. (If this were true, then we could put everything together to conclude that all stationary points are saddle points, for a harmonic oscillator.) This, however, is not always possible, due to the boundary conditions $\xi(t_1) = \xi(t_2) = 0$. If ξ is to change from zero to a large value, then $\dot{\xi}$ may also have to be large, if the time interval is short enough. Problem 6 deals quantitatively with this issue. For now, let's just say that in some cases S_0 is a minimum, and in some cases S_0 is a saddle point. "Least action", therefore, is a misnomer.

4. It is sometimes claimed that nature has a "purpose", in that it seeks to take the path that produces the minimum action. In view of the above remark, this is incorrect. In fact, nature does the exact opposite. It takes every path, treating them all on equal footing. We simply end up seeing the path with the least action, due to the way the quantum mechanical phases add.

   It would be a harsh requirement, indeed, to demand that nature make a "global" decision (that is, to compare paths that are separated by large distances), and to choose the one with the smallest action. Instead, we see that everything takes place on a "local" scale. Nearby phases simply add, and everything works out automatically.

   Of course, why things should work according to this general plan of adding phases is a whole different matter.

5. Consider a function, $f(x)$, of one variable (for ease of terminology). Let $f(b)$ be a local minimum of f. There are two basic properties of this minimum. The first is that $f(b)$ is smaller than all nearby values. The second is that the slope of f is zero at b. From the above remarks, we see that (when dealing with the action, S) the first

property is completely irrelevant, and the second one is the whole point. Therefore, saddle points (and maxima, although we showed above that these never exist for S) are just as good as minima, as far as the constructive addition of the $e^{iS/\hbar}$ phases is concerned.

6. Of course, given that classical mechanics is the approximate theory, while quantum mechanics is the (more) correct one, it is quite silly to justify the principle of stationary action by demonstrating its equivalence with $F = ma$; we should be doing it the other way around. However, because your intuition is based on $F = ma$, I'll assume that it's easier to start with $F = ma$ as the given fact, rather than calling upon the latent quantum-mechanics intuition hidden deep within all of us. Maybe someday.

   At any rate, in more advanced theories dealing with fundamental issues concerning the building blocks of matter (where the action is of the same order of magnitude as $\hbar$) the approximate $F = ma$ theory is invalid, and you *have* to use the Lagrangian method. ♣

5.3 Forces of constraint

One nice thing about the Lagrangian method is that we are free to impose the given constraints at the beginning of the problem, thereby immediately reducing the number of variables. This is always done (perhaps without thinking) whenever a particle is constrained to move on a wire or surface, etc. Often we are not concerned with the exact nature of the forces doing the constraining, but only with the resulting motion, given that the constraints hold. By imposing the constraints at the outset, we can find this motion, but we can't say anything about the constraining forces.

If we want to determine these constraining forces, we must take a different approach. A major point, as we will show, is that we must not impose the constraints too soon. This, of course, leaves us with a larger number of variables to deal with, so the calculations are more cumbersome. But the benefit is that we are able to find the constraining forces.

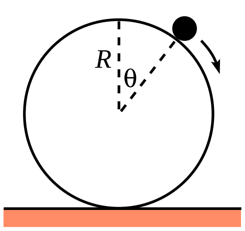

Figure 5.3

Consider the example of a particle sliding off a fixed frictionless sphere of radius R (see Fig. 5.3). Let's say that we are concerned only with finding the equation of motion for θ, and not the constraining forces. Then we can write everything in terms of θ, because we know that the radial distance, r, is constrained to be R. The kinetic energy is $mR^2\dot{\theta}^2/2$, and the potential energy (relative to the center of the sphere) is $mgr\cos\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta, \tag{5.24}$$

and the equation of motion, via eq. (5.3), is

$$\ddot{\theta} = (g/R)\sin\theta. \tag{5.25}$$

(This is simply the tangential $F = ma$ statement.)

Now let's say we want to find the constraining normal force that the sphere applies to the particle. To do this, let's solve the problem in a different way and write things in terms of both r and θ. Also (and here's the critical step), let's be really picky and say that r isn't *exactly* constrained to be R, because in the real world the particle actually pushes into the sphere a little bit. This may seem a bit silly, but it's really the whole point. The particle pushes in a (very tiny) distance until the sphere gets squashed enough to push back with the appropriate force to keep the particle from pressing in any more. (Just consider the sphere to be made of lots of little springs with very large spring constants.) The particle is therefore subject to a (very) steep potential due to the sphere. The constraining potential, $V(r)$, looks something like the plot in Fig. 5.4.

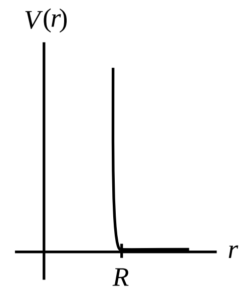

Figure 5.4

The *true* Lagrangian for the system is thus

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta - V(r). \tag{5.26}$$

(The $\dot{r}^2$ term in the kinetic energy will turn out to be insignificant.) The equations of motion from varying θ and r are therefore

$$\begin{aligned} mr^2\ddot{\theta} &= mgr\sin\theta, \\ m\ddot{r} &= mr\dot{\theta}^2 - mg\cos\theta - V'(r). \end{aligned} \tag{5.27}$$

Having written down the equations of motion, we will *now* apply the constraint condition that $r = R$. This condition implies $\dot{r} = \ddot{r} = 0$. (Of course, r isn't *really* equal to R, but any differences are inconsequential from this point onward.) Our first equation then simply gives eq. (5.25), while the second yields

$$-\frac{dV}{dr}\bigg|_{r=R} = mg\cos\theta - mR\dot{\theta}^2. \tag{5.28}$$

But $F_c \equiv -dV/dr$ is the constraint force applied in the r direction, which is precisely the force we are looking for. The normal force of constraint is therefore

$$F_c(\theta, \dot{\theta}) = mg\cos\theta - mR\dot{\theta}^2. \tag{5.29}$$

(This is simply the radial $F = ma$ statement.) This result is valid only if $F_c > 0$. The particle leaves the sphere if F_c becomes equal to zero.

REMARKS:

1. What if we instead had (unwisely) chosen our coordinates to be x and y, instead of r and θ? Since the distance from the particle to the sphere is $\eta \equiv \sqrt{x^2 + y^2} - R$, we obtain a true Lagrangian of

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(\eta). \tag{5.30}$$

The equations of motion are (using the chain rule)

$$m\ddot{x} = -\frac{dV}{d\eta}\frac{\partial\eta}{\partial x}, \qquad \text{and} \qquad m\ddot{y} = -mg - \frac{dV}{d\eta}\frac{\partial\eta}{\partial y}. \tag{5.31}$$

We now apply the constraint condition $\eta = 0$. Since $-dV/d\eta$ equals the constraint force F_c, you can show that the equations at our disposal (namely, the two E-L equations and the constraint equation) are

$$m\ddot{x} = F_c\frac{x}{R}, \qquad m\ddot{y} = -mg + F_c\frac{y}{R}, \qquad \text{and} \qquad \sqrt{x^2 + y^2} - R = 0. \tag{5.32}$$

These three equations are (in principal) sufficient to determine the three unknowns ($\ddot{x}$, $\ddot{y}$, and F_c) as functions of the quantities x, $\dot{x}$, y, and $\dot{y}$.

2. You can see from eqs. (5.27) and (5.32) that the E-L equations end up taking the form,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + F_c \frac{\partial \eta}{\partial q_i}, \tag{5.33}$$
for each coordinate, q_i. Here η is the constraint equation of the form $\eta = 0$. In the r,θ coordinates, $\eta = r - R$. And in the x,y coordinates, $\eta = \sqrt{x^2+y^2} - R$. The set of E-L equations, combined with the $\eta = 0$ condition, will give you exactly the number of equations needed to determine all the unknowns (the $\ddot{q}_i$ and F_c) in terms of the q_i and $\dot{q}_i$.

3. When trying to determine the forces of constraint, you can simply start with eqs. (5.33) (without bothering to write down $V(\eta)$), but you must be careful to make sure that η does indeed represent the distance the particle is from where it should be. In the r,θ coordinates above, if someone gives you the constraint condition as $7(r - R) = 0$, and if you use the left-hand-side of this as the η in eq. (5.33), then you will get the wrong constraint force. Likewise, in the x,y coordinates, writing the constraint as $y - R\sin(\arccos(x/R)) = 0$ would give you the wrong force.

   The best way to avoid this problem is, of course, to pick one of your variables as the distance the particle is from where it should be (or at least a linear function of the distance, as in the case of the "r" above). ♣

5.4 Change of coordinates

When L is written in terms of cartesian coordinates, x,y,z we have shown in Section 5.1 that the Euler-Lagrange equations are exactly the same as Newton's $F = ma$ equations (see eq. (5.6)). But what about the case when we use polar, spherical, or other coordinates? The equivalence of the *E-L* equations and Newton's laws is not obvious. As far as trusting the *E-L* equations for such coordinates goes, you can achieve peace-of-mind in two ways. You can accept the principle of stationary action as something so beautiful and so profound that it simply has to work for any choice of coordinates. Or, you can take the more mundane road and show through a change of coordinates that if the *E-L* equations hold for one set of coordinates (and we know they *do* hold for cartesian coordinates), then they also hold for any other coordinates (of a certain form, described below). We will demonstrate this fact through the explicit change of coordinates. This calculation is straightforward but a little messy, so you may want to skip this section and just settle for the 'beautiful and profound' reasoning.

Consider the set of coordinates,
$$x_i : \quad (x_1, x_2, \ldots, x_N). \tag{5.34}$$
For example, x_1,x_2,x_3 could be the cartesian x,y,z coordinates of one particle, and x_4,x_5,x_6 could be the r,θ,ϕ polar coordinates of a second particle, and so on. Assume that the *E-L* equations hold for these variables, that is,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \qquad (1 \le i \le N). \tag{5.35}$$

We know that there is at least one set of variables for which this is true, namely the cartesian coordinates. Consider a new set of variables which are functions of the x_i and t,

$$q_i = q_i(x_1, x_2, \ldots, x_N; t). \tag{5.36}$$

We will restrict ourselves to the case where the q_i do not depend on the $\dot{x}_i$. (This is quite reasonable. If the coordinates depended on the velocities, then we wouldn't be able to label points in space with definite coordinates. We'd have to worry about how the particles were behaving when they were at the points. These would be strange coordinates indeed.) Note that we can, in theory, invert eqs. (5.36) and express the x_i as functions of the q_i,

$$x_i = x_i(q_1, q_2, \ldots, q_N; t). \tag{5.37}$$

Claim 5.2 *If eq. (5.35) is true for the x_i coordinates, and if the x_i and q_i are related by eqs. (5.37), then eq. (5.35) is also true for the q_i coordinates. That is,*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}, \qquad (1 \le m \le N). \tag{5.38}$$

Proof: We have

$$\frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_m}. \tag{5.39}$$

(If the x_i depended on the $\dot{q}_i$, we would have the additional term $\sum(\partial L/\partial x_i)(\partial x_i/\partial \dot{q}_m)$, but we have excluded such dependence.) Let's rewrite the $\partial \dot{x}_i/\partial \dot{q}_m$ term. From eq. (5.37), we have

$$\dot{x}_i = \sum_{m=1}^{N} \frac{\partial x_i}{\partial q_m}\dot{q}_m + \frac{\partial x_i}{\partial t}. \tag{5.40}$$

Therefore,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m}. \tag{5.41}$$

Substituting this into eq. (5.39) and taking the time derivative of both sides gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right). \tag{5.42}$$

In the second term here, it is legal to switch the order of the d/dt and $\partial/\partial q_m$ derivatives.

REMARK: Let's prove that this switching is legal, just is case you have your doubts.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right) &= \sum_{k=1}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_m}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial q_m}\right) \\ &= \frac{\partial}{\partial q_m}\left(\sum_{k=1}^{N} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right) \\ &= \frac{\partial}{\partial q_m}\dot{x}_i, \end{aligned} \tag{5.43}$$

as was to be shown. The switching would work even if the $\partial x_i/\partial q_m$ were functions of the $\dot{q}_i$. ♣

In the first term in eq. (5.42), we can use the given information in eq. (5.35) and rewrite the $(d/dt)(\partial L/\partial \dot{x}_i)$ term. We obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) &= \sum_{i=1}^{N} \frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_m} \\ &= \frac{\partial L}{\partial q_m}, \end{aligned} \tag{5.44}$$

as was to be shown. ■

Therefore, if the Euler-Lagrange equations are true for a set of coordinates, x_i (and they *are* true for cartesian coordinates), then they are also true for any other set of coordinates, q_i, satisfying eq. (5.36). For those of you who look at the principle of stationary action with distrust (thinking that it might be a coordinate-dependent statement), this proof should put you at ease. The Euler-Lagrange equations are truly equivalent to $F = ma$ in any coordinates.

Note that the above proof did not in any way use the precise form of L. If L were equal to $T + V$, or $7T + \pi V^2/T$, or any other arbitrary function, our result would still be true: If eqs. (5.35) are true for one set of coordinates, then they are also true for any coordinates q_i satisfying eqs. (5.36). The point is that the only L for which the hypothesis is true at all (that is, for which eq. (5.35) holds) is $L = T - V$.

REMARK: On one hand, it is quite amazing how little we assumed in proving the above claim. *Any* new coordinates of the very general form (5.36) will satisfy the *E-L* equations, as long as the original coordinates do. If the *E-L* equations had, say, a factor of 5 on the right-hand side of eqs. (5.35), then they would *not* hold in arbitrary coordinates. (To see this, just follow the proof through with the factor of 5.)

On the other hand, the claim seems quite obvious, if you make an analogy with a function instead of a functional. Consider the function $f(z) = z^2$. This has a minimum at $z = 0$, consistent with the fact that $df/dz = 0$ at $z = 0$. Let's instead write f in terms of the variable defined by, say, $z = y^4$. Then $f(y) = y^8$, and f has a minimum at $y = 0$, consistent with the fact that df/dy equals zero at $y = 0$. So $f' = 0$ holds in both coordinates at the corresponding points $y = z = 0$. This is the (simplified) analog of the *E-L* equations holding in both coordinates. In both cases, the derivative equation describes where the stationary value occurs.

This change-of-variables result may be stated in a more geometrical (and friendly) way. If you plot a function and then stretch the horizontal axis in an arbitrary manner, a stationary value will still be a stationary value after the stretching. (A picture is worth a dozen equations, it appears.)

As an example of an equation that does *not* hold for all coordinates, consider the preceding example, but with $f' = 1$ instead of $f' = 0$. In terms of z, $f' = 1$ when $z = 1/2$. But in terms of y, $f' = 1$ when $y = (1/8)^{1/7}$. The points $z = 1/2$ and $y = (1/8)^{1/7}$ are not the same point. In other words, $f' = 1$ is not a coordinate-independent statement. Most equations, of course, are coordinate dependent. The special thing about $f' = 0$ is that a stationary point is a stationary point no matter how you look at it.[7] ♣

[7]However, a stationary point in one coordinate system might be located at a kink in another

5.5 Conservation Laws

5.5.1 Cyclic coordinates

Consider the case where the Lagrangian does not depend on a certain coordinate q_k. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0. \tag{5.45}$$

Therefore

$$\frac{\partial L}{\partial \dot{q}_k} = C, \tag{5.46}$$

where C is some constant, independent of time. In this case, we say that q_k is a *cyclic* coordinate, and that $\partial L/\partial \dot{q}_k$ is a *conserved* quantity (since it doesn't change with time).

If cartesian coordinates are used, then $\partial L/\partial \dot{x}_k$ is simply the momentum $m\dot{x}_k$, because $\dot{x}_k$ appears in only the $m\dot{x}_k^2/2$ term (we exclude cases where V depends on $\dot{x}_k$). We therefore call $\partial L/\partial \dot{q}_k$ the *generalized momentum* corresponding to the coordinate q_k. And in cases where $\partial L/\partial \dot{q}_k$ does not change with time, we call it a *conserved momentum.*

Note that a generalized momentum need not have the units of linear momentum, as the angular-momentum examples below show.

Example 1: Linear momentum

Consider a ball thrown through the air. Considering the full three dimensions, the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz. \tag{5.47}$$

There is no x or y dependence here, so both $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial \dot{y} = m\dot{y}$ are constants, as we well know.

Example 2: Angular momentum in polar coordinates

Consider a potential which depends only on the distance to the source. (Examples of such potentials are the gravity and electrostatic ones, which go like $1/r$; and also the spring potential, which goes like $(r-a)^2$, where a is the equilibrium length.) In polar coordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{5.48}$$

There is no θ dependence here, so $\partial L/\partial \dot{\theta} = mr^2\dot{\theta}$ is a constant. Since $r\dot{\theta}$ is the speed in the tangential direction, $mr(r\dot{\theta})$ is the angular momentum, which we see is conserved.

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coordinate system, so that f' is not defined there. For example, if we had said that $z = y^{1/4}$, then $f(y) = y^{1/2}$, which has an undefined slope at $y = 0$. But let's not worry about this.

Example 3: Angular momentum in spherical coordinates

In spherical coordinates, consider a potential that depends only on r and θ. (Our convention for spherical coordinates will be that θ is the angle down from the north pole, and ϕ is the angle around the equator.) The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r,\theta). \tag{5.49}$$

There is no ϕ dependence here, so $\partial L/\partial\dot{\phi} = mr^2\sin^2\theta\dot{\phi}$ is a constant. Since $r\sin\theta\dot{\phi}$ is the speed in the tangential direction around the z-axis and $r\sin\theta$ is the distance from the z-axis, $m(r\sin\theta)(r\sin\theta\dot{\phi})$ is the angular momentum about the z-axis, which we see is conserved.

5.5.2 Energy conservation

We will now derive another conservation law, that of energy. The conservation of momentum or angular momentum above arose when the Lagrangian was independent of x, θ, or ϕ. Conservation of energy arises when the Lagrangian is independent of time. This conservation law is different from those in the above momenta examples, because t is not a coordinate which the stationary-action principle can be applied to, as x, θ, and ϕ are. (The *E-L* equation, eq. (5.3), makes no sense if x is replaced by t.) Therefore, we're going to have to prove this conservation law in a different way. Consider the quantity

$$E = \left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - L. \tag{5.50}$$

E will (usually) turn out to be the energy. We'll show this below.

Claim 5.3 *If L has no explicit time dependence (that is, $\partial L/\partial t = 0$), then E is conserved (that is, $dE/dt = 0$).*

(Note that there is one partial derivative and one full derivative in this statement.)
Proof: L is a function of the q_i, the $\dot{q}_i$, and possibly t. Making copious use of the chain rule, we have

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i\right) - \frac{dL}{dt}\\ &= \sum_{i=1}^{N}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right) - \left(\sum_{i=1}^{N}\left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right) + \frac{\partial L}{\partial t}\right). \end{aligned} \tag{5.51}$$

There are five terms here. The second cancels with the fourth. And the first (after using the E-L equation, eq. (5.3), to rewrite it) cancels with the third. We therefore arrive at the simple result,

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}. \tag{5.52}$$

In the event that $\partial L/\partial t = 0$ (that is, there are no t's sitting on the paper when you write down L), which will invariably be the case in the situations we consider (we won't consider potentials that depend on time), we have $dE/dt = 0$. ■

Not too many things are constant with respect to time, and the quantity E has units of energy, so it's a good bet that it is the energy. Let's show this in cartesian coordinates. (However, see the remark below.)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z), \tag{5.53}$$

so eq. (5.50) gives

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z), \tag{5.54}$$

which is, of course, the total energy.

To be sure, taking the kinetic energy T and subtracting the potential energy V to obtain L, and then using eq. (5.50) to produce $E = T + V$, seems like a rather convoluted way to arrive at $T + V$. The point of all this is that we used the E-L equations to *prove* that E is conserved. Although we know very well from the $F = ma$ methods in Chapter 4 that the sum $T + V$ is conserved, it's not fair to assume that it is conserved in our new Lagrangian formalism. We have to show that this *follows* from the *E-L* equations.

EXAMPLE: The quantity E in eq. (5.50) gives the energy of the system only if the entire system is represented by the Lagrangian. That is, the Lagrangian must represent a closed system (with no external forces). If the system is not closed, then eq. (5.52) and Claim 5.3 are still perfectly valid, but the quantity E may not be the energy of the system. Problem 9 is a good example of such a situation.

Another simple example is a mass in the x-y plane that is subjected to an external force which gives an acceleration g in the negative y direction. If we assume that the mass starts at rest, then $\dot{y} = -gt$. The Lagrangian is therefore $L = m\dot{x}^2/2 + m(gt)^2/2$, and so eq. (5.50) gives $E = m\dot{x}^2/2 - m(gt)^2/2$, which is not the energy. Of course, we can get rid of the external force by instead saying that the particle moves under the influence of the potential $V(y) = mgy$. The Lagrangian of this closed system is then $L = m(\dot{x}^2 + \dot{y})^2/2 - mgy$, and so eq. (5.50) gives $E = m(\dot{x}^2 + \dot{y})^2/2 + mgy$, which is indeed the energy of the mass.

At any rate, most of the systems you will deal with are closed, so you can generally ignore this remark and assume that the E in eq. (5.50) gives the energy. ♣

5.6 Noether's Theorem

We now present one of the most wonderful and useful theorems in physics. It deals with two fundamental concepts in physics, namely *symmetry* and *conserved quantities*. The theorem may be stated as follows.

Theorem 5.4 *(***Noether's Theorem***) For each symmetry of the Lagrangian, there is a conserved quantity.*

(By "symmetry", we mean that if the coordinates are changed by some small quantities, then the Lagrangian has no first order change in these quantities. By "conserved quantity", we mean a quantity that does not change in time. The result in the previous section for cyclic coordinates is a special case of this theorem.)

Proof: Let the Lagrangian be invariant (to first order in the small number ϵ) under the change of coordinates,

$$q_i \longrightarrow q_i + \epsilon K_i(q). \tag{5.55}$$

Each $K_i(q)$ may be a function of all the q_i, which we collectively denote by the shorthand, q.

EXAMPLE: Consider a mass on a spring (with zero equilibrium length), in the x-y plane. $L = (m/2)(\dot{x}^2 + \dot{y}^2) + (k/2)(x^2 + y^2)$ is invariant under the change of coordinates, $x \to x + \epsilon y$, $y \to y - \epsilon x$ (to first order in ϵ, as you can check). So in this case, $K_x = y$ and $K_y = -x$.

Of course, someone else might come along with $K_x = 5y$ and $K_y = -5x$, which is also a symmetry. And indeed, any factor can be taken out of ϵ and put into the K_i's without changing the quantity $\epsilon K_i(q)$ in eq. (5.55). Any such modification will simply bring an overall constant factor into our eventual conserved quantity (which will also be conserved, of course), so it is irrelevant. ♣

The fact that the Lagrangian does not change at first order in ϵ means that

$$\begin{aligned} 0 = \frac{dL}{d\epsilon} &= \sum_i \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon} \right) \\ &= \sum_i \left(\frac{\partial L}{\partial q_i} K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i \right). \end{aligned} \tag{5.56}$$

Using the E-L equation, we may rewrite this as

$$\begin{aligned} 0 &= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i \right) \\ &= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} K_i \right). \end{aligned} \tag{5.57}$$

Therefore, the quantity

$$P(q, \dot{q}) \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i(q) \tag{5.58}$$

is conserved with respect to time. It is given the generic name of *conserved momentum*. (But it need not have the units of linear momentum). ■

As Noether most keenly observed
(And for which much acclaim is deserved),
For each symmetry,
We can easily see
That a quantity must be conserved.

Example 1:

In the mass-on-spring example mentioned above, with $L = (m/2)(\dot{x}^2+\dot{y}^2)+(k/2)(x^2+y^2)$, we have $K_x = y$ and $K_y = -x$. So the conserved momentum is

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m(\dot{x}y - \dot{y}x). \tag{5.59}$$

This is simply the z-component of the angular momentum. (The angular momentum is conserved here because the potential $V(x, y) = x^2 + y^2 = r^2$ depends only on the distance from the origin; we'll discuss such potentials in Chapter 6).

Example 2:

Consider a thrown ball. We have $L = (m/2)(\dot{x}^2 + \dot{y}^2) - mgy$. This is clearly invariant under translations in x, that is, $x \to x + \epsilon$ (x is a cyclic coordinate). (We only need invariance to first order in ϵ, but this is clearly invariant to all orders.) Therefore, $K_x = 1$ and $K_y = 0$. (Of course, K_x may be chosen to be any constant, but we may as well pick it to be 1.) So the conserved momentum is

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m\dot{x}. \tag{5.60}$$

This is simply the x-component of the linear momentum (as we saw in Example 1 in Section 5.5.1).

Example 3:

Let $L = (m/2)(5\dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{y}^2) + C(2x - y)$. This is clearly invariant under the transformation $x \to x + \epsilon$ and $y \to y + 2\epsilon$. Therefore, $K_x = 1$ and $K_y = 2$. So the conserved momentum is

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m(5\dot{x}-\dot{y})(1) + m(-\dot{x}+2\dot{y})(2) = m(3\dot{x}+3\dot{y}). \tag{5.61}$$

The overall factor of $3m$ doesn't matter, of course.

REMARKS:

1. Note that in some cases the K_i's are functions of the coordinates, and in some cases they are not.

2. In simple systems, as in Example 2 above, it is quite obvious why the resulting P is conserved. But in more complicated systems, as in Example 3 above (which has an L of the type that arises in Atwood's machine problems; see the problems for this chapter), the resulting P might not have an obvious interpretation. But at least you know that it is conserved, and this will invariably help in solving a problem.

3. Although conserved quantities are extremely useful in studying a physical situation, it should be stressed that there is no more information contained in the them than there is in the E-L equations. Conserved quantities are simply the result of integrating the E-L equations. For example, if you write down the E-L equations for the third example above, and then add the 'x' equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) to twice the '$y$' equation (which is $-m\ddot{x} + 2m\ddot{y} = -C$), you find $3m(\ddot{x} + \ddot{y}) = 0$, as desired.

Of course, you might have to do some guesswork to find the proper combination of the E-L equations that gives a zero on the right-hand side. But you'd have to do some guesswork anyway, to find the symmetry for Noether's theorem.

At any rate, a conserved quantity is useful because it is an integrated form of the E-L equations. It puts you one step closer to solving the problem, compared to where you would be if you started with the second-order E-L equations.

4. Does every system have a conserved momentum? Certainly not. The one-dimensional problem of a falling ball ($m\ddot{z} = -mg$) doesn't have one. And if you write down an arbitrary potential in 3-D, odds are there won't be one. In a sense, things have to contrive nicely for there to be a conserved momentum. In some problems, you can just look at the physical system and see what the symmetry is, but in others (for example, in some of the Atwood's-machine problems for this chapter), the symmetry is not at all obvious.

5. By "conserved quantity", we mean a quantity that depends on (at most) the coordinates and their first derivatives (that is, not on their second derivatives). If we do not make this restriction, then it is trivial to construct quantities that do not vary with time. For example, in the third example above, the 'x' E-L equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) tells us that $5m\ddot{x} - m\ddot{y}$ has its time derivative equal to zero. Note that an equivalent way of excluding these trivial quantities is to say that the value of a conserved quantity depends on initial conditions (that is, velocities and positions). The quantity $5m\ddot{x} - m\ddot{y}$ does not satisfy this criterion, because its value is always constrained to be $2C$. ♣

5.7 Small oscillations

In many physical systems, a particle may undergo small oscillations around an equilibrium point. In Section 4.2, we showed that the frequency of these small oscillations is

$$\omega = \sqrt{\frac{V''(x_0)}{m}}\,, \tag{5.62}$$

where $V(x)$ is the potential energy, and x_0 is the equilibrium point.

However, this result holds only for *one-dimensional* motion (we will see below why this is true). In more complicated systems, such as the one described below, it is necessary to use another procedure to obtain the frequency ω. This procedure is a fail-proof one. It is applicable in all situations. It is, however, a bit more involved than simply writing down eq. (5.62). So in 1-D problems, eq. (5.62) is what you want to use.

We'll demonstrate our fail-proof method through the following problem.

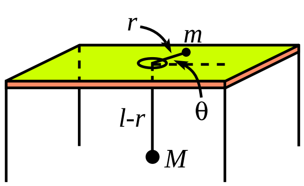

Figure 5.5

Problem:

A mass m is free to move on a frictionless table and is connected by a string, which passes through a hole in the table, to a mass M which hangs below (see Fig. 5.5). Assume M moves in a vertical line only, and assume the string always remains taut.

(a) Find the equations of motion for the variables r and θ shown in the figure.

(b) Under what condition does m undergo circular motion?

(c) What is the frequency of small oscillations (in the variable r) about this circular motion?

Solution:

(a) Let the string have length ℓ (this length won't matter). Then the Lagrangian is (we'll call it '$\mathcal{L}$' here, and save 'L' for the angular momentum, which arises below)

$$\mathcal{L} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mg(\ell - r). \tag{5.63}$$

We've taken the table to be at height zero, for the purposes of potential energy, but any other value could be chosen, of course. The equations of motion obtained from varying θ and r are

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0, \\ (M+m)\ddot{r} &= mr\dot{\theta}^2 - Mg. \end{aligned} \tag{5.64}$$

The first equation says that angular momentum is conserved (much more about this in later chapters). The second equation says that the Mg gravitational force accounts for the acceleration of the two masses along the direction of the string, plus the centripetal acceleration of m.

(b) The first of eqs. (5.64) says that $mr^2\dot{\theta} = L$, where L is some constant (the angular momentum) which depends on the initial conditions. Plugging $\dot{\theta} = L/mr^2$ into the second of eqs. (5.64) gives

$$(M+m)\ddot{r} = \frac{L^2}{mr^3} - Mg. \tag{5.65}$$

Circular motion occurs when $\dot{r} = \ddot{r} = 0$. Therefore, the radius of the circular orbit is given by

$$r_0^3 = \frac{L^2}{Mmg}. \tag{5.66}$$

REMARK: Note that since $L = mr^2\dot{\theta}$, eq. (5.66) is equivalent to

$$mr_0\dot{\theta}^2 = Mg, \tag{5.67}$$

which can be obtained by simply letting $\ddot{r} = 0$ in the second of eqs. (5.64). In other words, the gravitational force on M exactly accounts for the centripetal acceleration of m. Given r_0, this equation determines what $\dot{\theta}$ must be (in order to have circular motion), and vice versa. ♣

(c) To find the frequency of small oscillations about a circular motion, we need to look at what happens to r if we perturb it slightly from its equilibrium value, r_0. Our fail-proof procedure is the following.

Let $r(t) \equiv r_0 + \delta(t)$ (where $\delta(t)$ is very small; more precisely, $\delta(t) \ll r_0$), and expand eq. (5.65) to first order in $\delta(t)$. Using

$$\frac{1}{r^3} \equiv \frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.68}$$

we have

$$(M+m)\ddot{\delta} = \frac{L^2}{mr_0^3}\left(1 - \frac{3\delta}{r_0}\right) - Mg. \tag{5.69}$$

The terms not involving δ on the right-hand side cancel, by the definition of r_0 (eq. (5.66)). (This cancellation will always occur in such a problem at this stage, due to the definition of the equilibrium point.) We are therefore left with

$$\ddot{\delta} + \left(\frac{3L^2}{(M+m)mr_0^4}\right)\delta = 0. \tag{5.70}$$

This is a nice simple-harmonic-oscillator equation in the variable δ. Therefore, the frequency of small oscillations about a circle of radius r_0 is

$$\omega = \sqrt{\frac{3L^2}{(M+m)mr_0^4}} = \sqrt{\frac{3M}{M+m}}\sqrt{\frac{g}{r_0}}, \tag{5.71}$$

where we have used eq. (5.66) to eliminate L in the second expression.

REMARKS: Let's look at some limits. For a given r_0, if $m \gg M$, then $\omega \approx \sqrt{3Mg/mr_0} \approx 0$. This makes sense (everything will be moving very slowly). Note that this frequency is equal to $\sqrt{3}$ times the frequency of the circular motion (which is $\sqrt{Mg/mr_0}$, from eq. (5.67)).

For a given r_0, if $m \ll M$, then $\omega \approx \sqrt{3g/r_0}$. This is not so obvious.

Note that the frequency of small oscillations is equal to the frequency of the circular motion if $M = 2m$. This condition is independent of r_0. ♣

The above procedure for finding the frequency of small oscillations may be summed up in three steps: (1) Find the equations of motion, (2) Find the equilibrium point, and (3) Let $x(t) \equiv x_0 + \delta(t)$ (where x_0 is the equilibrium point of the relevant variable), and expand one of the equations of motion (or a combination of them) to first order in δ, to obtain a simple-harmonic-oscillator equation for δ.

REMARK: Note that if you simply used the potential energy in the above problem (which is Mgr, up to a constant) in eq. (5.62), then you would obtain a frequency of zero, which is incorrect.

You *can* use eq. (5.62) to find the frequency, if you instead use the "effective potential" for this problem (namely $L^2/(2mr^2) + Mgr$), and if you use the total mass, $M+m$, as the mass in eq. (5.62). The reason for this will become clear in Chapter 6 when we introduce the effective potential.

In many problems, however, it is not obvious what "modified potential" should be used, or what mass should be used in eq. (5.62), so it is generally much safer to take a deep breath and go through an expansion similar to the one in part (c) above. ♣

Note that the one-dimensional result in eq. (5.62) is, of course, simply a special case of our above expansion procedure. We can repeat the derivation of Section 4.2 in the present language. In one dimension, we have $m\ddot{x} = -V'(x)$. Let x_0 be the equilibrium point (so that $V'(x_0) = 0$), and let $x \equiv x_0 + \delta$. Expanding $m\ddot{x} = -V'(x)$ to first order in δ, we have $m\ddot{\delta} = -V'(x_0) - V''(x_0)\delta - \cdots$. Hence, $m\ddot{\delta} \approx -V''(x_0)\delta$, as desired.

5.8 Other applications

The formalism developed in section 5.2 works for *any* function $L(x, \dot{x}, t)$. If our goal is to find the stationary points of $S \equiv \int L$, then eq. (5.13) holds, no matter what L is. There is no need for L to be equal to $T - V$, or indeed, to have anything to do with physics. And t need not have anything to do with time. All that is required is that the quantity x depends on the parameter t, and that L depends on only x, $\dot{x}$, and t (and not, for example, on $\ddot{x}$; see Problem *****). The formalism is very general and quite powerful, as the following problem demonstrates.

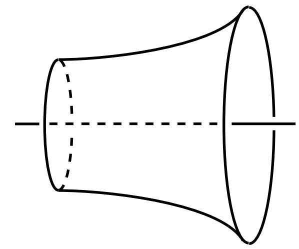

Figure 5.6

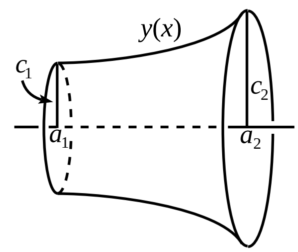

Figure 5.7

Example (Minimal surface of revolution): A surface of revolution has two given rings as its boundary (see Fig. 5.6). What should the shape of the surface be so that it has the minimum possible area?

(We'll present two solutions. A third solution is left for Problem 24.)

First solution: Let the surface be generated by rotating the curve $y = y(x)$ around the x-axis. The boundary conditions are $y(a_1) = c_1$ and $y(a_2) = c_2$ (see Fig. 5.7). Slicing the surface up into vertical rings, we see that the area is given by

$$A = \int_{a_1}^{a_2} 2\pi y \sqrt{1 + y'^2}\, dx. \tag{5.72}$$

The goal is to find the function $y(x)$ that minimizes this integral. We therefore have exactly the same situation as in Section 5.2, except that t has now become x, and x has become y. Our 'Lagrangian' is thus $L \propto y\sqrt{1 + y'^2}$.

We must now apply the E-L equation to this Lagrangian. We've relegated this rather tedious calculation to Lemma 5.5 at the end of this section. Eq. (5.81) gives (with $f(y) = y$ here)

$$1 + y'^2 = By^2. \tag{5.73}$$

At this point we can either cleverly guess that the solution is

$$y(x) = \frac{1}{b}\cosh b(x + d) \tag{5.74}$$

(where $b = \sqrt{B}$, and d is a constant of integration), or we can separate variables to obtain $dy/\sqrt{By^2 - 1} = dx$, and then integrate this to obtain the same result.

Therefore, the answer to our problem is that $y(x)$ takes the form of eq. (5.74), with b and d determined by the boundary conditions

$$c_1 = \frac{1}{b}\cosh b(a_1 + d), \qquad \text{and} \qquad c_2 = \frac{1}{b}\cosh b(a_2 + d). \tag{5.75}$$

In the nice case where $c_1 = c_2$, we know that the minimum occurs in the middle, so we may choose $d = 0$ and $a_1 = -a_2$.

Remark: Solutions for b and d exist only for certain ranges of the a's and c's. Basically, if $a_2 - a_1$ is too big, then there is no solution. In this case, the minimal 'surface' turns out to be the two given circles, attached by a line (which isn't a nice two-dimensional surface). If you perform an experiment with soap bubbles (which want to minimize their area), and

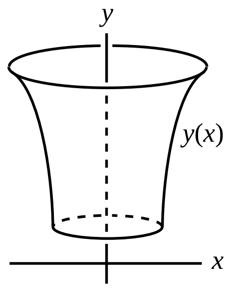

Figure 5.8

if you pull the rings too far apart, then the surface will break and disappear, as it tries to form the two circles. Problem 27 deals with this issue. ♣

Second solution: Consider the surface to be obtained by rotating a curve around the y-axis, instead of the x-axis (see Fig. 5.8). The area is then given by

$$A = \int_{a_1}^{a_2} 2\pi x\sqrt{1+y'^2}\,dx. \tag{5.76}$$

(The function $y(x)$ may be double-valued, so it may not really be a function. But it looks like a function locally, and all of our formalism deals with local variations.) Our 'Lagrangian' is now $L \propto x\sqrt{1+y'^2}$, and the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0. \tag{5.77}$$

(The nice thing about this solution is the '0' on the right-hand side, which arose from the fact that L does not depend on y.) Therefore, $xy'/\sqrt{1+y'^2}$ is constant, and we have

$$y' = \frac{1}{\sqrt{(bx)^2-1}}, \tag{5.78}$$

for some constant b. Since the integral of $1/\sqrt{z^2-1}$ is $\cosh^{-1} z$, we find $y(x) = (1/b)\cosh^{-1}(bx) - d$. Inverting, we obtain

$$x(y) = \frac{1}{b}\cosh b(y+d), \tag{5.79}$$

which is the same as eq. (5.74), with x and y interchanged.

Numerous other 'extremum' problems are solvable with these general techniques. A few are presented in the problems for this chapter.

Let us now prove the following Lemma, which we invoked in the first solution above. This Lemma is very useful, because it is common to encounter problems where the quantity to be extremized depends on the arclength, $\sqrt{1+y'^2}$, and takes the form $\int f(y)\sqrt{1+y'^2}\,dx$.

We'll give two proofs. The first proof uses the Euler-Lagrange equation. The calculation here gets a bit messy, so it's a good idea to work through it once and for all — it's not something you'd want to repeat too often. The second proof makes use of a conserved quantity. And in contrast to the first proof, this method is exceedingly clean and simple. It actually *is* something you'd want to repeat quite often. (But we'll still do it once and for all.)

Lemma 5.5 *Let $f(y)$ be a given function of y. Then the function $y(x)$ that extremizes the integral,*

$$\int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\,dx, \tag{5.80}$$

satisfies the differential equation,

$$1 + y'^2 = Bf(y)^2, \tag{5.81}$$

where B is a constant of integration.[8]

First Proof: The goal is to find the function $y(x)$ that extremizes the integral in eq. (5.80). We therefore have exactly the same situation as in section 5.2, except with x in place of t, and y in place of x. Our 'Lagrangian' is thus $L = f(y)\sqrt{1+y'^2}$, and the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(f \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}}\right) = f'\sqrt{1+y'^2}, \tag{5.82}$$

where $f' \equiv df/dy$. We must now perform some straightforward (albeit tedious) differentiations. Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, gives

$$\frac{f'y'^2}{\sqrt{1+y'^2}} + \frac{fy''}{\sqrt{1+y'^2}} - \frac{fy'^2y''}{(1+y'^2)^{3/2}} = f'\sqrt{1+y'^2}. \tag{5.83}$$

Multiplying through by $(1+y'^2)^{3/2}$ and simplifying gives

$$fy'' = f'(1+y'^2). \tag{5.84}$$

We have completed the first step of the solution, namely producing a differential equation. Now we must integrate it. Eq. (5.84) happens to be integrable for arbitrary functions $f(y)$. If we multiply through by y' and rearrange, we obtain

$$\frac{y'y''}{1+y'^2} = \frac{f'y'}{f}. \tag{5.85}$$

Taking the dx integral of both sides gives $(1/2)\ln(1+y'^2) = \ln(f) + C$, where C is an integration constant. Exponentiation then gives (with $B \equiv e^{2C}$)

$$1 + y'^2 = Bf(y)^2, \tag{5.86}$$

as was to be shown. The next task would be to solve for y', and to then separate variables and integrate. But we would need to be given a specific function $f(y)$ to be able do this.

Second Proof: Note that our 'Lagrangian', $L = f(y)\sqrt{1+y'^2}$, is independent of x. Therefore, in analogy with the conserved energy given in eq. (5.50), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{-f(y)}{\sqrt{1+y'^2}} \tag{5.87}$$

is independent of x. Call it $1/\sqrt{B}$. Then we have easily reproduced eq. (5.86). ■

IMPORTANT REMARK: As demonstrated by the brevity of the second solution here, it is highly advantageous to make use of the conserved quantity E (which arose from independence of x). ♣

[8] B, along with one other constant of integration, will eventually be determined from the boundary conditions, once eq. (5.81) is integrated to solve for y.

5.9 Exercises

Section 5.2: The principle of stationary action

1. **Explicit minimization** $*$

   A ball is thrown upward. Let $y(t)$ be the height as a function of time, and assume $y(0) = 0$ and $y(T) = L$. Guess a solution for y of the form $y(t) = a_0 + a_1 t + a_2 t^2$, and explicitly calculate the action between $t = 0$ and $t = L$. Show that the action is minimized when $a_2 = -g/2$. (This gets slightly messy.)

2. **$\ddot{x}$ dependence** $**$

   Let there be $\ddot{x}$ dependence (in addition to x,$\dot{x}$,t dependence) in the Lagrangian in Theorem 5.1. There will then be the additional term $(\partial L/\partial \ddot{x}_a)\ddot{\beta}$ in eq. (5.17). It is tempting to integrate this term by parts twice, and then arrive at a modified form of eq. (5.19):

$$\frac{\partial L}{\partial x_0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}_0}\right) = 0. \tag{5.88}$$

   Is this a valid result? If not, where is the error in its derivation?

Section 5.3: Forces of constraint

3. **Constraint on a circle**

   A bead slides with speed v around a horizontal loop of radius R. What force does the loop apply to the bead? (Ignore gravity.)

4. **Constraint on a curve** $**$

   Let the horizontal plane be the x-y plane. A bead slides with speed v along a curve described by the function $y = f(x)$. What force does the curve apply to the bead? (Ignore gravity.)

5.10 Problems

Section 5.1: The Euler-Lagrange equations

1. **Moving plane** **

   A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released (see Fig. 5.9). What is the horizontal acceleration of the plane?

   (This problem was posed in Chapter 2. If you haven't already done so, try solving it using $F = ma$. You will then have a much greater appreciation for the Lagrangian method.)

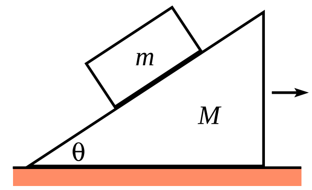

Figure 5.9

2. **Two masses, one swinging** ***

   Two equal masses m, connected by a string, hang over two pulleys (of negligible size), as shown in Fig. 5.10. The left one moves in a vertical line, but the right one is free to swing back and forth (in the plane of the masses and pulleys). Find the equations of motion.

   Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude ϵ (with $\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?

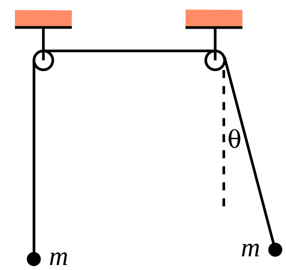

Figure 5.10

3. **Falling sticks** **

   Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other, as in Fig. 5.11. The bottom end of the lower stick is hinged at the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. At the instant they are released, what are the angular accelerations of the two sticks? (You may work in the approximation where ϵ is very small).

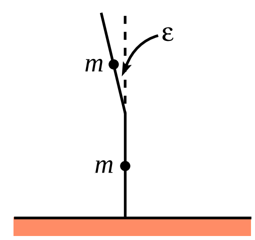

Figure 5.11

4. **Pendulum with oscillating support** **

   A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$ (see Fig. 5.12). Find the equation of motion for the angle of the pendulum.

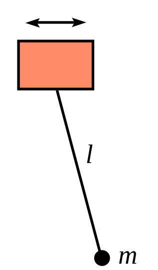

Figure 5.12

5. **Inverted pendulum** ****

   (a) A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates vertically with a position given by $y(t) = A\cos(\omega t)$ (see Fig. 5.13). Find the equation of motion for the angle of the pendulum (measured relative to its upside-down position).

   (b) It turns out that if ω is large enough, then if the stick is initially nearly upside-down, it will, surprisingly, *not* fall over as time goes by. Instead,

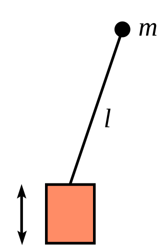

Figure 5.13

it will (sort of) oscillate like an upside-down pendulum. (This can be shown numerically.) Give a qualitative argument why the stick doesn't fall over. You don't need to be exact; just make the result believable.

Section 5.2: The principle of stationary action

6. **Minimum or saddle** $**$

In eq. (5.23), let $t_1 = 0$ and $t_2 = T$, for convenience. And let the $\xi(t)$ be an easy-to-deal-with "triangular" function, of the form

$$\xi(t) = \begin{cases} \epsilon t/T, & 0 \le t \le T/2, \\ \epsilon(1 - t/T), & T/2 \le t \le T. \end{cases} \tag{5.89}$$

Under what conditions is the ΔS in eq. (5.23) negative?

Section 5.3: Forces of constraint

7. **Mass on plane** $**$

A mass m slides down a frictionless plane which is inclined at an angle θ. Show that the normal force from the plane is the familiar $mg\cos\theta$.

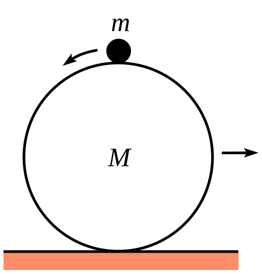

Figure 5.14

8. **Leaving the moving sphere** $***$

A particle of mass m sits on top of a frictionless sphere of mass M (see Fig. 5.14). The sphere is free to slide on the frictionless ground. The particle is given an infinitesimal kick. Let θ be the angle which the radius to the particle makes with the vertical. Find the equation of motion for θ. Also, find the force of constraint in terms of θ and $\dot\theta$.

Section 5.5: Conservation Laws

9. **Bead on stick** $*$

A stick is pivoted at the origin and swings around in a horizontal plane at constant angular speed ω. A bead of mass m slides frictionlessly along the stick. Let r be the radial position of the bead. Find the conserved quantity E given in eq. (5.50). Explain why this quantity is *not* the energy of the bead.

Section 5.6: Noether's Theorem

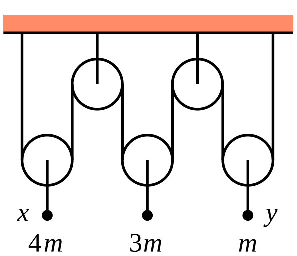

Figure 5.15

10. **Atwood's machine 1** $**$

Consider the Atwood's machine shown in Fig. 5.15. The masses are $4m$, $3m$, and m. Let x and y be the heights of the left and right masses (relative to their initial positions). Find the conserved momentum.

11. **Atwood's machine 2** $**$

Consider the Atwood's machine shown in Fig. 5.16. The masses are $5m$, $4m$, and $2m$. Let x and y be the heights of the left two masses (relative to their initial positions). Find the conserved momentum.

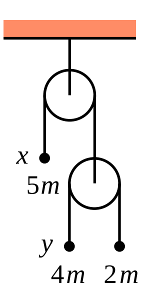

Figure 5.16

Section 5.7: Small oscillations

12. **Pulley pendulum** $**$

    A mass M is attached to a massless hoop (of radius R) which lies in a vertical plane. The hoop is free to rotate about its fixed center. M is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string (see Fig. 5.17). Find the equation of motion for the angle through which the hoop rotates. What is the frequency of small oscillations? (You may assume $M > m$.)

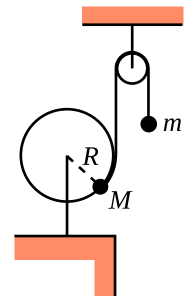

Figure 5.17

13. **Three hanging masses** $***$

    A mass M is fixed at the midpoint if a long string, at the ends of which are tied masses m. The string hangs over two frictionless pulleys (located at the same height), as shown in Fig. 5.18. The pulleys are a distance $2l$ apart and have negligible size. Assume that the mass M is constrained to move in a vertical line midway between the pulleys.

    Let θ be the angle the string from M makes with the horizontal. Find the equation of motion for θ. (This is a bit messy.) Find the frequency of small oscillations around the equilibrium point. (You may assume $M < 2m$.)

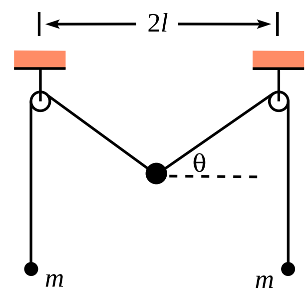

Figure 5.18

14. **Bead on rotating hoop** $**$

    A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (see Fig. 5.19). Find the equation of motion for the position of the bead. What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium?

    There is one value of ω that is rather special. What is it, and why is it special?

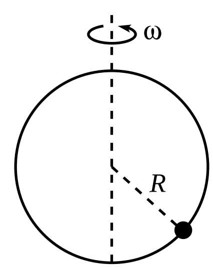

Figure 5.19

15. **Another bead on rotating hoop** $**$

    A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point (see Fig. 5.20). Find the equation of motion for the position of the bead. Also, find the frequency of small oscillations about the equilibrium point.

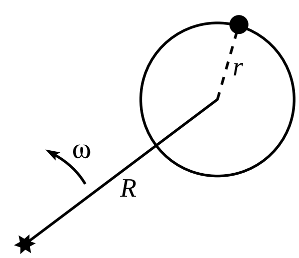

Figure 5.20

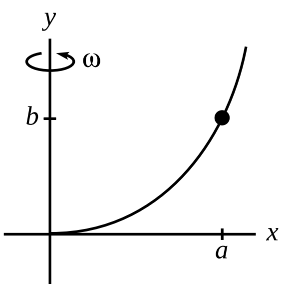

Figure 5.21

16. **Rotating curve** ***

The curve $y = f(x) = b(x/a)^\lambda$ is rotated around the y-axis with constant frequency ω. A bead moves without friction along the curve (see Fig. 5.21). Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist? (This gets a little messy.)

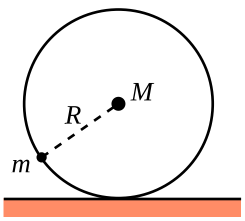

Figure 5.22

17. **Mass on wheel** **

A mass m is fixed to a given point at the edge of a wheel of radius R. The wheel is massless, except for a mass M located at its center (see Fig. 5.22). The wheel rolls without slipping on a horizontal table. Find the equation of motion for the angle through which the wheel rolls. For the case where the wheel undergoes small oscillations, find the frequency.

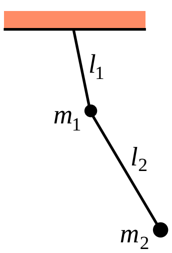

Figure 5.23

18. **Double pendulum** ****

Consider a double pendulum made of two masses, m_1 and m_2, and two rods of lengths ℓ_1 and ℓ_2 (see Fig. 5.23). Find the equations of motion.

For small oscillations, find the normal modes and their frequencies for the special case $\ell_1 = \ell_2$ (and check the limits $m_1 \gg m_2$ and $m_1 \ll m_2$). Do the same for the special case $m_1 = m_2$ (and check the limits $\ell_1 \gg \ell_2$ and $\ell_1 \ll \ell_2$).

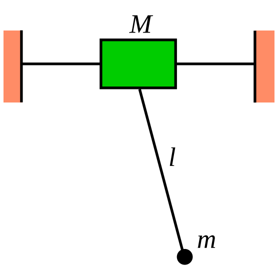

Figure 5.24

19. **Pendulum with free support** **

A pendulum of mass m and length ℓ is hung from a support of mass M which is free to move horizontally on a frictionless rail (see Fig. 5.24). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

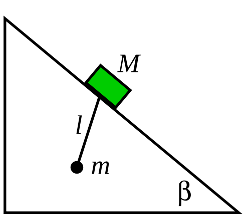

Figure 5.25

20. **Pendulum support on inclined plane** **

A mass M slides down a frictionless plane inclined at angle β. A pendulum, with length ℓ and mass m, is attached to M (see Fig. 5.25). Find the equations of motion, and also the normal modes for small oscillations.

21. **Tilting plane** ∗∗∗

A mass M is fixed at the right-angled vertex where a massless rod of length ℓ is connected to a very long massless rod (see Fig. 5.26). A mass m is free to move frictionlessly along the long rod. The rod of length ℓ is hinged at a support, and the whole system is free to rotate, in the plane of the rods, about the support.

Let θ be the angle of rotation of the system, and let x be the distance between m and M. Find the equations of motion. Find the normal modes when θ and x are both very small.

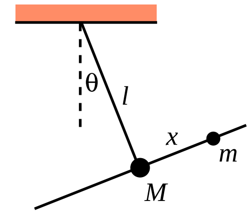

Figure 5.26

22. **Motion on a cone** ∗∗∗

A particle moves on a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The cone has a half-angle equal to α (see Fig. 5.27). Find the equations of motion.

Let the particle move in a circle of radius r_0. What is the frequency, ω, of this circular motion? Let the particle be perturbed slightly from this motion. What is the frequency, Ω, of the oscillations about the radius r_0? Under what conditions does $\Omega = \omega$?

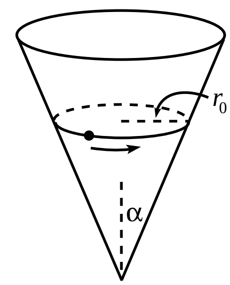

Figure 5.27

Section 5.8: Other applications

23. **Shortest distance in a plane**

In the spirit of section 5.8, show that the shortest path between two points in a plane is a straight line.

24. **Minimal surface** ∗∗

Derive the shape of the minimal surface discussed in Section 5.8, by demanding that a cross-sectional 'ring' (that is, the region between the planes $x = x_1$ and $x = x_2$) is in equilibrium; see Fig. 5.28. *Hint:* The tension must be constant throughout the surface.

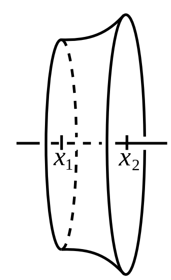

Figure 5.28

25. **The brachistochrone** ∗∗∗

A bead is released from rest and slides down a frictionless wire that connects the origin to a given point, as shown in Fig. 5.29. You wish to shape the wire so that the bead reaches the endpoint in the shortest possible time.

Let the desired curve be described by the function $y(x)$, with downward being the positive y direction, for convenience.

(a) Show that $y(x)$ satisfies

$$-2yy'' = 1 + y'^2, \tag{5.90}$$

and that a first integral of the motion is

$$1 + y'^2 = \frac{C}{y}. \tag{5.91}$$

where C is a constant.

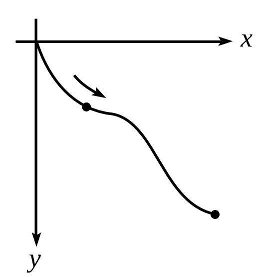

Figure 5.29

(b) Show that x and y may be written as

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta). \tag{5.92}$$

You may do this by simply verifying that they satisfy eq. (5.91). But try to do it also by solving the differential equation from scratch. (Eq. (5.92) is the parametrization of a *cycloid*, which is the path taken by a point on the edge of a rolling wheel.)

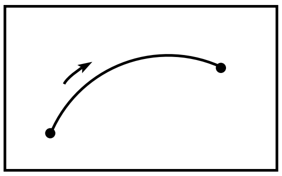

Figure 5.30

26. **Index of refraction** $**$

Assume that the speed of light in a given slab of material is proportional to the height above the base of the slab.[9] Show that light moves in circular arcs in this material; see Fig. 5.30. You may assume that light takes the path of shortest time between two points (Fermat's principle of least time).

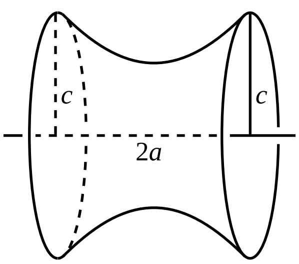

Figure 5.31

27. **Existence of minimal surface** $**$

Consider the minimal surface from Section 5.8. Consider the special case where the two rings have the same radius (that is, $c_1 = c_2 \equiv c$). Let $2a \equiv a_1 - a_2$ be the distance between the rings (see Fig. 5.31).

What is the largest value of a/c for which a minimal surface exists? (You will have to solve something numerically here.)

[9]In other words, the index of refraction of the material, n, as a function of the height, y, is given by $n(y) = y_0/y$, where y_0 is some length that is larger than the height of the slab.

5.11 Solutions

1. **Moving plane**

Let x_1 be the horizontal coordinate of the plane (with positive x_1 to the right). Let x_2 be the horizontal coordinate of the mass (with positive x_2 to the left). (See Fig. 5.32.) Then it is easy to see that the height fallen by the mass is $\Delta y = (x_1 + x_2)\tan\theta$. The Lagrangian is therefore

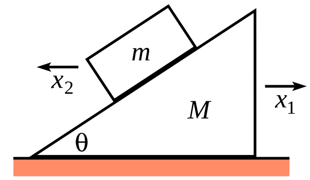

Figure 5.32

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\Big(\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2\tan^2\theta\Big) + mg(x_1 + x_2)\tan\theta. \tag{5.93}$$

The equations of motion from varying x_1 and x_2 are

$$\begin{aligned} M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta, \\ m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta. \end{aligned} \tag{5.94}$$

Note that these two equations immediately yield conservation of momentum, $M\ddot{x}_1 = m\ddot{x}_2$. We may easily solve for $\ddot{x}_1$ to obtain

$$\ddot{x}_1 = \frac{mg\tan\theta}{M(1+\tan^2\theta) + m\tan^2\theta}. \tag{5.95}$$

REMARKS: For given M and m, the angle θ_0 which maximizes $\ddot{x}_1$ is found to be

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}. \tag{5.96}$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, we have $\ddot{x}_1 \approx g/\tan\theta$. This makes sense, because m falls essentially straight down, and the plane gets squeezed out to the right.

In the limit $M \gg m$, we have $\ddot{x}_1 \approx g(m/M)\tan\theta/(1+\tan^2\theta) = g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $\ddot{x}_2 = (M/m)\ddot{x}_1 \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of $\ddot{x}_2$ implies that the acceleration of m *along* the plane is essentially equal to $\ddot{x}_2/\cos\theta \approx g\sin\theta$, as expected. ♣

2. **Two masses, one swinging**

Let r be the distance from the swinging mass to the pulley, and let θ be the angle of the swinging mass. Then the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr + mgr\cos\theta. \tag{5.97}$$

The equations of motion from varying r and θ are

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - g(1-\cos\theta), \\ \frac{d}{dt}(r^2\dot{\theta}) &= -gr\sin\theta. \end{aligned} \tag{5.98}$$

The first equation deals with the acceleration along the direction of the string. The second equation equates the torque from gravity with change in angular momentum.

If we do a (coarse) small-angle approximation and keep only terms up to first order in θ, we find at $t = 0$ (using the initial condition, $\dot{r} = 0$)

$$\begin{aligned} \ddot{r} &= 0, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \tag{5.99}$$

These say that the left mass stays still, and the right mass behaves just like a pendulum.

If we want to find the leading term in the initial acceleration of the left mass (i.e., the leading term in $\ddot{r}$), we need to be a little less coarse in our approximation. So let's keep terms in eq. (5.98) up to second order in θ. We then have at $t = 0$ (using the initial condition, $\dot{r} = 0$)

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - \frac{1}{2}g\theta^2, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \tag{5.100}$$

The second equation says that the right mass undergoes harmonic motion. In this problem, it is given that the amplitude is ϵ. So we have

$$\theta(t) = \epsilon \cos(\omega t + \phi), \tag{5.101}$$

where $\omega = \sqrt{g/r}$. Plugging this into the first equation gives

$$2\ddot{r} = \epsilon^2 g \left(\sin^2(\omega t + \phi) - \frac{1}{2}\cos^2(\omega t + \phi) \right). \tag{5.102}$$

If we average over a few periods, both $\sin^2\alpha$ and $\cos^2\alpha$ average to $1/2$, so we find

$$\ddot{r}_{\text{avg}} = \frac{\epsilon^2 g}{8}. \tag{5.103}$$

This is a small second-order effect. It is positive, so the left mass slowly begins to climb.

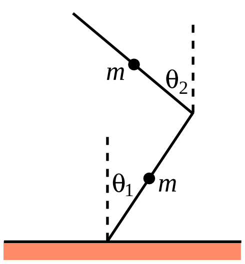

Figure 5.33

3. **Falling sticks**

Let $\theta_1(t)$ and $\theta_2(t)$ be defined as in Fig. 5.33. Then the position of the bottom mass in cartesian coordinates is $(r\sin\theta_1, r\cos\theta_1)$, and the position of the top mass is $(2r\sin\theta_1 - r\sin\theta_2, 2r\cos\theta_1 + r\cos\theta_2)$. So the potential energy of the system is

$$V(\theta_1, \theta_2) = mgr(3\cos\theta_1 + \cos\theta_2). \tag{5.104}$$

The kinetic energy is somewhat more complicated. The K.E. of the bottom mass is simply $mr^2\dot{\theta}_1^2/2$. The K.E. of the top mass is

$$\frac{1}{2}mr^2\Big((2\cos\theta_1\dot{\theta}_1 - \cos\theta_2\dot{\theta}_2)^2 + (-2\sin\theta_1\dot{\theta}_1 - \sin\theta_2\dot{\theta}_2)^2\Big). \tag{5.105}$$

Let's now simplify this, using the small-angle approximations. The terms involving $\sin\theta$ will be fourth order in the small θ's, so we may neglect them. Also, we may approximate $\cos\theta$ by 1, since we will have dropped only terms of at least fourth order. So this K.E. turns into $(1/2)mr^2(2\dot{\theta}_1 - \dot{\theta}_2)^2$. In other words, the masses move essentially horizontally. Therefore, using the small-angle approximation $\cos\theta \approx 1 - \theta^2/2$ to rewrite V, we have

$$L \approx \frac{1}{2}mr^2\left(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) - mgr\left(4 - \frac{3}{2}\theta_1^2 - \frac{1}{2}\theta_2^2\right). \tag{5.106}$$

The equations of motion from θ_1 and θ_2 are, respectively,

$$\begin{aligned} 5\ddot{\theta}_1 - 2\ddot{\theta}_2 &= \frac{3g}{r}\theta_1 \\ -2\ddot{\theta}_1 + \ddot{\theta}_2 &= \frac{g}{r}\theta_2. \end{aligned} \tag{5.107}$$

At the instant the sticks are released, $\theta_1 = 0$ and $\theta_2 = \epsilon$. Solving our two equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ gives

$$\ddot{\theta}_1 = \frac{2g}{r}\epsilon, \qquad \text{and} \qquad \ddot{\theta}_2 = \frac{5g}{r}\epsilon. \tag{5.108}$$

4. **Pendulum with oscillating support**

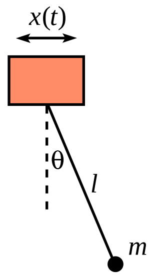

Figure 5.34

Let θ be defined as in Fig. 5.34. With $x(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X,Y)_m = (x + \ell\sin\theta, -\ell\cos\theta). \tag{5.109}$$

The square of its speed is

$$V_m^2 = \ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta. \tag{5.110}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.111}$$

The equation of motion from varying θ is

$$\ell\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta. \tag{5.112}$$

Plugging in the explicit form of $x(t)$, we have

$$\ell\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0. \tag{5.113}$$

This makes sense. Someone in the frame of the support (which has horizontal acceleration $\ddot{x} = -A\omega^2\cos(\omega t)$) may as well be living in a world where the acceleration from gravity has a component g downward and a component $A\omega^2\cos(\omega t)$ to the right. Eq. (5.120) is simply the equation for the force in the tangential direction.

A small-angle approximation gives

$$\ddot{\theta} + \omega_0^2\theta = a\omega^2\cos(\omega t), \tag{5.114}$$

where $\omega_0 \equiv \sqrt{g/\ell}$ and $a \equiv A/\ell$. This equation is simply that of a driven oscillator, which we solved in Chapter 3. The solution is

$$\theta(t) = \frac{a\omega^2}{\omega_0^2 - \omega^2}\cos(\omega t) + C\cos(\omega_0 t + \phi), \tag{5.115}$$

where C and ϕ are determined by the initial conditions.

If ω happens to equal ω_0, then the amplitude goes to infinity. But in the real world, there is a damping term which keeps the coefficient of the $\cos(\omega t)$ term finite.

5. **Inverted pendulum**

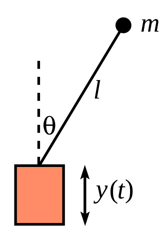

Figure 5.35

(a) Let θ be defined as in Fig. 5.35. With $y(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X,Y)_m = (\ell\sin\theta, y + \ell\cos\theta). \tag{5.116}$$

The square of its speed is

$$V_m^2 = \ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta. \tag{5.117}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta). \tag{5.118}$$

The equation of motion is

$$\ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta. \qquad (5.119)$$

Plugging in the explicit form of $y(t)$, we have

$$\ell\ddot{\theta} + \sin\theta(A\omega^2\cos(\omega t) - g) = 0. \qquad (5.120)$$

This makes sense. Someone in the frame of the support (which has acceleration $\ddot{y} = -A\omega^2\cos(\omega t)$) may as well be living in a world where the acceleration from gravity is $(g - A\omega^2\cos(\omega t))$ downward. Eq. (5.120) is simply the equation for the force in the tangential direction.

Assuming θ is small, we may set $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big) = 0. \qquad (5.121)$$

where $\omega_0 \equiv \sqrt{g/\ell}$ and $a \equiv A/\ell$.

(b) Eq. (5.121) cannot be solved exactly, but we can get a general idea of how θ depends on time in two different ways.

One way is to solve the equation numerically. Figs. [numerical] show the results with parameters $\ell = 1$m, $A = 0.1$m, and $g = 10\text{m/s}^2$. In the first plot, $\omega = 10\text{s}^{-1}$; in the second plot, $\omega = 100\text{s}^{-1}$. The stick falls over in first case, but undergoes oscillatory motion in the second case. Apparently, if ω is large enough, the stick will not fall over.

Now let's explain this phenomenon in a second way: by making rough, order-of-magnitude arguments. At first glance, it seems surprising that the stick will stay up. It seems like the average of the 'effective gravity' acceleration (averaged over a few periods of the ω oscillations) in eq. (5.121) is $(-\theta g)$, since the $\cos(\omega t)$ term averages to zero (or so it appears). So one might think that there is a net downward force, making the stick fall over.

The fallacy in this reasoning is that the average of the $\theta\cos(\omega t)$ term is *not* zero, because θ undergoes tiny oscillations with frequency ω (as seen in Fig. [numerical]). The values of θ and $\cos(\omega t)$ are correlated; the θ at the t when $\cos(\omega t) = 1$ is larger than the θ at the t when $\cos(\omega t) = -1$. So there is a net positive contribution to the $\theta\cos(\omega t)$ part of the force. (And, indeed, it may be large enough to keep the pendulum up, as we show below.) This reasoning is enough to make the phenomenon believable (at least to me), but let's do a little more.

How large is this positive contribution from the $\theta\cos(\omega t)$ term? Let's make some rough approximations. We will look at the case where ω is large and $a \equiv A/\ell$ is small. (more precisely, we will assume $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons seen below). Look at one of the little oscillations (with frequency ω in Fig. [pendulum]). The average position of the pendulum doesn't change much over one of these small periods, so we can look for an approximate solution to eq. (5.120) of the form

$$\theta(t) \approx C + b\cos(\omega t), \qquad (5.122)$$

where $b \ll C$. C will change over time, but on the scale of $1/\omega$ it is essentially constant.

Plugging this guess for θ into eq. (5.121), we find, at leading order (using $a \ll 1$ and $a\omega^2 \gg \omega_0^2$), $-\omega^2 b\cos(\omega t) + Ca\omega^2\cos(\omega t) = 0$. So we must have $b = aC$.

Our approximate solution for θ, on short time scales, is therefore

$$\theta \approx C\Big(1 + a\cos(\omega t)\Big). \tag{5.123}$$

From eq. (5.121), the average acceleration of θ, over a period $T = 2\pi/\omega$, is then

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= \overline{-C\Big(1 + a\cos(\omega t)\Big)\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= -C\Big(a^2\omega^2\overline{\cos^2(\omega t)} - \omega_0^2\Big) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&\equiv -C\Omega^2.
\end{aligned} \tag{5.124}$$

But from eq. (5.122) we see that the average acceleration of θ is simply $\ddot{C}$. So we have

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0. \tag{5.125}$$

Therefore, C oscillates sinusoidally with frequency

$$\Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}. \tag{5.126}$$

We must have $a\omega > \sqrt{2}\omega_0$ if this frequency is to be real so that the pendulum stays up. Note that since we have assumed $a \ll 1$, we see that $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 \gg \omega_0^2$ (for the case where the pendulum stays up), which is consistent with our assumption above.

If $a\omega \gg \sqrt{g/\ell}$, then we have $\Omega \approx a\omega/\sqrt{2}$. Such is the case if the pendulum lies in a horizontal plane where the acceleration from gravity is zero.

6. **Minimum or saddle**

For the given $\xi(t)$, the integrand in eq. (5.23) is symmetric about the midpoint, so we obtain

$$\begin{aligned}
\Delta S &= \int_0^{T/2}\left(m\left(\frac{\epsilon}{T}\right)^2 - k\left(\frac{\epsilon t}{T}\right)^2\right)dt. \\
&= \frac{m\epsilon^2}{2T} - \frac{k\epsilon^2 T}{24}.
\end{aligned} \tag{5.127}$$

This is negative if $T > \sqrt{12m/k} \equiv 2\sqrt{3}/\omega$. Since the period of the oscillator is $\tau \equiv 2\pi/\omega$, we see that $T \equiv t_2 - t_1$ must be greater than $(\sqrt{3}/\pi)\tau$ in order for ΔS to be negative (provided that we are using our triangular function for ξ).

Roughly speaking, if $T \gtrsim \tau$, then the stationary point of S is a saddle point. And if $T \lesssim \tau$, then the stationary point of S is a minimum. In the latter case, the basic point is that T is small enough so that there is no way for ξ to get large, without making $\dot{\xi}$ large also.

Remark: Our triangular function for ξ was easy to deal with, but it is undoubtedly not the function that gives the best chance of making ΔS negative. In other words, we should expect that it is possible to make T (slightly) less than $(\sqrt{3}/\pi)\tau$, and still be able to find a function ξ that makes ΔS negative. So let's try to find the smallest possible T for which ΔS can be negative.

It turns out that a function with the shape $\xi(t) \propto \sin(\pi t/T)$ gives the best chance of making ΔS negative.[10] With this form of ξ, we find $\Delta S \propto m\pi^2/T - kT$. This is negative if $T > \pi\sqrt{m/k} \equiv \pi/\omega$. In other words, if $T > \tau/2$, then the stationary value of S is a saddle point. And if $T < \tau/2$, then the stationary value of S is a minimum.

Our $T > (\sqrt{3}/\pi)\tau$ approximation using the triangular function was therefore a fairly good one. ♣

7. **Mass on plane**

First Solution: The most convenient coordinates in this problem are w and z, where w is the distance upward along the plane, and z is the distance perpendicularly away from it. The Lagrangian is then

$$\frac{1}{2}m(\dot{w}^2 + \dot{z}^2) - mg(w\sin\theta + z\cos\theta) - V(z), \tag{5.129}$$

where $V(z)$ is the (very steep) constraining potential. The two equations of motion are

$$\begin{aligned} m\ddot{w} &= -mg\sin\theta, \\ m\ddot{z} &= -mg\cos\theta - \frac{dV}{dz}. \end{aligned} \tag{5.130}$$

At this point we invoke the constraint $z = 0$. So $\ddot{z} = 0$, and the second equation gives us

$$F_c \equiv -V'(0) = mg\cos\theta, \tag{5.131}$$

as desired. We also obtain the usual result of $\ddot{w} = -g\sin\theta$.

Second Solution: We can also solve this problem by using the horizontal and vertical components, x and y. We'll choose $(x, y) = (0, 0)$ to be at the top of the plane. The (very steep) constraining potential is $V(z)$, where $z \equiv x\sin\theta + y\cos\theta$ is the distance from the mass to the plane (as you can verify). The Lagrangian is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(z) \tag{5.132}$$

Keeping mind that $z \equiv x\sin\theta + y\cos\theta$, the two equations of motion are (using the chain rule)

$$\begin{aligned} m\ddot{x} &= -\frac{dV}{dz}\frac{\partial z}{\partial x} = -V'(z)\sin\theta, \\ m\ddot{y} &= -mg - \frac{dV}{dz}\frac{\partial z}{\partial y} = -mg - V'(z)\cos\theta. \end{aligned} \tag{5.133}$$

[10] You can show this by invoking a theorem from Fourier analysis which says that any function satisfying $\xi(0) = \xi(T) = 0$ can be written as the sum $\xi(t) = \sum_1^\infty c_n \sin(n\pi t/T)$, where the c_n are numerical coefficients. When this sum is plugged into eq. (5.23), you can show that all the cross terms (terms involving two different values of n) integrate to zero. Using the fact that the average value of $\sin^2\theta$ and $\cos^2\theta$ is $1/2$, the rest of the integral is easily found to give

$$\Delta S = \frac{1}{4}\sum_1^\infty c_n^2 \left(\frac{m\pi^2 n^2}{T} - kT\right). \tag{5.128}$$

In order to obtain the smallest value of T that can make this negative, we clearly want only the $n = 1$ term to exist in the sum.

At this point we invoke the constraint condition $x = -y\cot\theta$ (that is, $z = 0$). This condition, along with the two E-L equations, allows us to solve for the three unknowns, $\ddot{x}$, $\ddot{y}$, and $V'(0)$. Using $\ddot{x} = -\ddot{y}\cot\theta$ in eqs. (5.133), we find

$$\ddot{y} = -g\sin^2\theta, \qquad \ddot{x} = g\cos\theta\sin\theta, \qquad F_c \equiv -V'(0) = mg\cos\theta. \tag{5.134}$$

The first two results here are simply the horizontal and vertical components of the acceleration along the plane.

8. **Leaving the moving sphere**

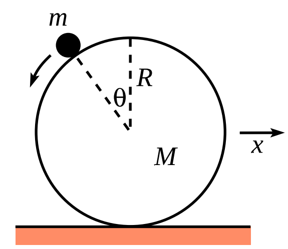

Figure 5.36

Let R be the radius of the sphere. Assume that the particle falls to the left and the sphere recoils to the right. Let θ be the angle from the top of the sphere (counterclockwise positive; see Fig. 5.36). Let x be the horizontal position of the sphere (positive to the right). Then the position of the particle (relative to the initial center of the sphere) is $(x - r\sin\theta, r\cos\theta)$, where r is constrained to be R. So the particle's velocity is $(\dot{x} - r\dot{\theta}\cos\theta, -r\dot{\theta}\sin\theta)$, and the square of its speed is $v_m^2 = \dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{x}\dot{\theta}\cos\theta$. We have ignored the negligible terms involving $\dot{r}$. To find the force of constraint, we have to consider the (very steep) potential, $V(r)$, keeping the particle on the sphere. The Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{x}\dot{\theta}\cos\theta) - mgr\cos\theta - V(r). \tag{5.135}$$

If $\dot{x} = 0$, this reduces to the previous problem. The equations of motion from varying x, θ, and r are

$$\begin{aligned} (M+m)\ddot{x} - mr\frac{d}{dt}(\dot{\theta}\cos\theta) &= 0, \\ r\ddot{\theta} - \ddot{x}\cos\theta &= g\sin\theta, \\ mr\dot{\theta}^2 - m\dot{x}\dot{\theta}\cos\theta - mg\cos\theta - V'(r) &= 0, \end{aligned} \tag{5.136}$$

where we have ignored the $\dot{r}$ terms. The first equation (when integrated) is conservation of momentum. The second equation is $F = ma$ for the θ direction in a world where 'gravity' pulls with strength g downward and strength $\ddot{x}$ to the left (in the accelerated frame of the sphere, this is what the particle feels). The third equation gives the radial force of constraint, $F = -dV/dr$ (evaluated at $r = R$), as

$$F(\theta, \dot{\theta}, \dot{x}) = mg\cos\theta + m\dot{x}\dot{\theta}\cos\theta - mR\dot{\theta}^2. \tag{5.137}$$

Let us now eliminate x from our equations. We may use the first equation in (5.136) to eliminate $\ddot{x}$ from the second equation. The result is (with $r = R$)

$$\ddot{\theta}(M + m\sin^2\theta) + m\dot{\theta}^2\cos\theta\sin\theta - \frac{g}{R}(M+m)\sin\theta = 0. \tag{5.138}$$

We may also use the integrated form of the first equation in (5.136) to eliminate $\dot{x}$ from the expression for F. (The integrated form is $(M+m)\dot{x} - mr\dot{\theta}\cos\theta = 0$, where the constant of integration is zero, due to the initial conditions, $\dot{x} = \dot{\theta} = 0$ at $t = 0$.) The result is (with $r = R$)

$$F(\theta, \dot{\theta}) = mg\cos\theta + \frac{m^2R}{M+m}\dot{\theta}^2\cos^2\theta - mR\dot{\theta}^2. \tag{5.139}$$

If $M \gg m$, these results agree with eq. (5.29), as they should, since the sphere will remain essentially at rest.

9. **Bead on stick**

The Lagrangian for the bead is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2. \tag{5.140}$$

Eq. (5.50) therefore gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2. \tag{5.141}$$

Claim 5.3 says that this quantity is conserved, because $\partial L/\partial t = 0$. But it is *not* the energy of the bead, due to the minus sign in the second term.

The main point here is that in order to keep the stick rotating at a constant angular speed, there must be an external force acting it. This force will cause work to be done on the bead, thereby changing its kinetic energy.

From the above equations, we see that $E = T - mr^2\omega^2$ is the quantity that doesn't change with time (where T is the kinetic energy).

10. **Atwood's machine 1**

First solution: If the left mass goes up by x, and the right mass goes up by y, then conservation of string says that the middle mass must go down by $x + y$. Therefore, the Lagrangian of the system is

$$\begin{aligned} L &= \frac{1}{2}(4m)\dot{x}^2 + \frac{1}{2}(3m)(-\dot{x}-\dot{y})^2 + \frac{1}{2}m\dot{y}^2 - \Big((4m)gx + (3m)g(-x-y) + mgy\Big) \\ &= \frac{7}{2}m\dot{x}^2 + 3m\dot{x}\dot{y} + 2m\dot{y}^2 - mg(x-2y). \end{aligned} \tag{5.142}$$

This is clearly invariant under the transformation $x \to x + 2\epsilon$ and $y \to y + \epsilon$. Hence, we can use Noether's theorem, with $K_x = 2$ and $K_y = 1$. So the conserved momentum is

$$P = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m(7\dot{x} + 3\dot{y})(2) + m(3\dot{x} + 4\dot{y})(1) = m(17\dot{x} + 10\dot{y}). \tag{5.143}$$

P is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $-(10/17)\dot{y}$.

Second solution: With x and y defined as in the first solution, the Euler-Lagrange equations are, from eq. (5.142),

$$\begin{aligned} 7m\ddot{x} + 3m\ddot{y} &= -mg, \\ 3m\ddot{x} + 4m\ddot{y} &= 2mg. \end{aligned} \tag{5.144}$$

Adding the second equation to twice the first gives

$$17m\ddot{x} + 10m\ddot{y} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(17m\dot{x} + 10m\dot{y}\Big) = 0. \tag{5.145}$$

Third solution: We can also solve the problem using only $F = ma$. Since the tension, T, is the same throughout the rope, we see that the three $F = dP/dt$ equations are

$$2T - 4mg = \frac{dP_{4m}}{dt}, \qquad 2T - 3mg = \frac{dP_{3m}}{dt}, \qquad 2T - mg = \frac{dP_m}{dt}. \tag{5.146}$$

The three forces depend on only two parameters, so there will be some combination of them that adds to zero. If we set $a(2T-4mg)+b(2T-3mg)+c(2T-mg)=0$, then $a+b+c=0$ and $4a+3b+c=0$, which is satisfied by $a=2$, $b=-3$, and $c=1$. Therefore (with x and y defined as in the first solution),

$$\begin{aligned} 0 &= \frac{d}{dt}(2P_{4m}-3P_{3m}+P_m) \\ &= \frac{d}{dt}\Big(2(4m)\dot{x}-3(3m)(-\dot{x}-\dot{y})+m\dot{y}\Big) \\ &= \frac{d}{dt}(17m\dot{x}+10m\dot{y}). \end{aligned} \tag{5.147}$$

11. **Atwood's machine 2**

First solution: The average of the heights of the right two masses (relative to their initial positions) is $-x$. Therefore, the position of the right mass must be $-2x-y$.

The Lagrangian of the system is thus

$$\begin{aligned} L &= \frac{1}{2}(5m)\dot{x}^2+\frac{1}{2}(4m)\dot{y}^2+\frac{1}{2}(2m)(-2\dot{x}-\dot{y})^2-\Big((5m)gx+(4m)gy+(2m)g(-2x-y)\Big) \\ &= \frac{13}{2}m\dot{x}^2+4m\dot{x}\dot{y}+3m\dot{y}^2-mg(x+2y). \end{aligned} \tag{5.148}$$

This is clearly invariant under the transformation $x \to x+2\epsilon$ and $y \to y-\epsilon$. Hence, we can use Noether's theorem, with $K_x=2$ and $K_y=-1$. So the conserved momentum is

$$P=\frac{\partial L}{\partial \dot{x}}K_x+\frac{\partial L}{\partial \dot{y}}K_y=m(13\dot{x}+4\dot{y})(2)+m(4\dot{x}+6\dot{y})(-1)=m(22\dot{x}+2\dot{y}). \tag{5.149}$$

P is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $-(1/11)\dot{y}$.

Second solution: With x and y defined as in the first solution, the Euler-Lagrange equations are, from eq. (5.148),

$$\begin{aligned} 13m\ddot{x}+4m\ddot{y} &= -mg, \\ 4m\ddot{x}+6m\ddot{y} &= -2mg. \end{aligned} \tag{5.150}$$

Subtracting half the second equation from the first gives

$$11m\ddot{x}+m\ddot{y}=0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(11m\dot{x}+m\dot{y}\Big)=0, \tag{5.151}$$

in agreement with (5.149).

Third solution: We can also solve the problem using only $F=ma$. Since the tension in the top rope is twice that in the bottom rope (because the net force on the massless lower pulley must be zero), we see that the three $F=dP/dt$ equations are

$$2T-5mg=\frac{dP_{5m}}{dt}, \qquad T-4mg=\frac{dP_{4m}}{dt}, \qquad T-2mg=\frac{dP_{2m}}{dt}. \tag{5.152}$$

The three forces depend on only two parameters, so there will be some combination of them that adds to zero. If we set $a(2T-5mg)+b(T-4mg)+c(T-2mg)=0$, then $2a + b + c = 0$ and $5a + 4b + 2c = 0$, which is satisfied by $a = 2$, $b = -1$, and $c = -3$. Therefore (with x and y defined as in the first solution),

$$\begin{aligned} 0 &= \frac{d}{dt}(2P_{5m} - P_{4m} - 3P_{2m}) \\ &= \frac{d}{dt}\Big(2(5m)\dot{x} - (4m)\dot{y} - 3(2m)(-2\dot{x} - \dot{y})\Big) \\ &= \frac{d}{dt}(22m\dot{x} + 2m\dot{y}). \end{aligned} \tag{5.153}$$

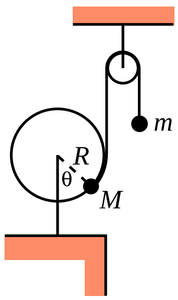

Figure 5.37

12. **Pulley pendulum**

Let the radius to M make an angle θ with the vertical (see Fig. 5.37). Then the coordinates of M are $R(\sin\theta, -\cos\theta)$. The height of the mass m, relative to its position when M is at the bottom of the hoop, is $y = -R\theta$. The Lagrangian is therefore (and yes, we've chosen a different $y = 0$ point for each mass, but such a definition only changes the potential by a constant amount, which is irrelevant)

$$L = \frac{1}{2}(M + m)R^2\dot{\theta}^2 + MgR\cos\theta + mgR\theta. \tag{5.154}$$

The equation of motion is

$$(M + m)R\ddot{\theta} = g(m - M\sin\theta). \tag{5.155}$$

This is, of course, just $F = ma$ along the direction of the string (since $Mg\sin\theta$ is the tangential component of the gravitational force on M).

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$, i.e., at the θ_0 for which $\sin\theta_0 = m/M$. Letting $\theta \equiv \theta_0 + \delta$, and expanding eq. (5.155) to first order in δ gives

$$\ddot{\delta} + \frac{Mg\cos\theta_0}{(M + m)R}\delta = 0. \tag{5.156}$$

So the frequency of small oscillations is

$$\omega = \sqrt{\frac{M\cos\theta_0}{M + m}}\sqrt{\frac{g}{R}}. \tag{5.157}$$

Remarks: If $M \gg m$, then $\theta_0 \approx 0$, and $\omega \approx \sqrt{g/R}$. This makes sense, because m can be ignored, and M essentially oscillates about the bottom of the hoop, just like a pendulum of length R.

If M is only slightly greater than m, then $\theta_0 \approx \pi/2$. So $\cos\theta_0 \approx 0$, and hence $\omega \approx 0$. This makes sense; the restoring force $g(m - M\sin\theta)$ does not change much as θ changes, so it's as if we have a pendulum in a weak gravitational field.

The frequency found in eq. (5.157) can actually be figured out with no calculations at all. Look at M at the equilibrium position. The tangential forces on it cancel, so all it feels is the $Mg\cos\theta_0$ normal force from the hoop which balances the radial component of the gravitational force. Therefore, for all the mass M knows, it is sitting at the bottom of a hoop of radius R in a world where gravity has strength $g' = g\cos\theta_0$. The general formula for the frequency of a pendulum is $\omega = \sqrt{F'/M'R}$, where F' is the downward force (which is Mg' here), and M' is the total mass being accelerated (which is $M + m$ here). This gives the ω in eq. (5.157). ♣

13. **Three hanging masses**

The height of M is $-l\tan\theta$. And the length of the string to M is $l/\cos\theta$, so the height of the m's is $l/\cos\theta$ (up to an additive constant). The speed of M is therefore $l\dot\theta/\cos^2\theta$, and the speed of the m's is $l\dot\theta\sin\theta/\cos^2\theta$. The Lagrangian is therefore

$$L = \frac{Ml^2}{2\cos^4\theta}\dot\theta^2 + \frac{ml^2\sin^2\theta}{\cos^4\theta}\dot\theta^2 + Mgl\tan\theta - \frac{2mgl}{\cos\theta}. \quad (5.158)$$

The equation of motion is

$$M\frac{d}{dt}\left(\frac{\dot\theta}{\cos^4\theta}\right) + 2m\frac{d}{dt}\left(\frac{\dot\theta\sin^2\theta}{\cos^4\theta}\right)$$
$$= \frac{M}{2}\frac{d}{d\theta}\left(\frac{\dot\theta^2}{\cos^4\theta}\right) + m\frac{d}{d\theta}\left(\frac{\sin^2\theta\dot\theta^2}{\cos^4\theta}\right) + \frac{Mg}{l\cos^2\theta} - \frac{2mg\sin\theta}{l\cos^2\theta}. \quad (5.159)$$

After a rather large amount of simplification, this becomes

$$M\ddot\theta\cos\theta + 2M\dot\theta^2\sin\theta + 2m\ddot\theta\cos\theta\sin^2\theta + 2m\dot\theta^2\cos^2\theta\sin\theta + 4m\dot\theta^2\sin^3\theta$$
$$= \frac{Mg}{l}\cos^3\theta - \frac{2mg}{l}\sin\theta\cos^3\theta. \quad (5.160)$$

Equilibrium occurs when $\ddot\theta = \dot\theta = 0$, and therefore when $\sin\theta_0 = M/2m$. Letting $\theta = \theta_0 + \delta$, we may expand eq. (5.160) to first order in δ. The terms involving $\dot\theta^2 = \dot\delta^2$ are of second order in δ and may be dropped. We find

$$\ddot\delta(M\cos\theta_0 + 2m\cos\theta_0\sin^2\theta_0)$$
$$= \delta(-3M\cos^2\theta_0\sin\theta_0 + 6m\cos^2\theta_0\sin^2\theta_0 - 2m\cos^4\theta_0)\frac{g}{l} \quad (5.161)$$

Plugging in $\sin\theta_0 = M/2m$, we find (after some simplification) that the frequency of small oscillations is

$$\omega^2 = \frac{g}{l}\left(\frac{(4m^2 - M^2)^{3/2}}{2mM(M+2m)}\right). \quad (5.162)$$

REMARK: In the limit where M is just slightly less than $2m$, we have $\omega \approx 0$. This makes sense, because the equilibrium position of M is very low, and the force changes slowly with position.

In the limit $M \to 0$, we have $\omega^2 \approx (g/l)(2m/M) \to \infty$. This makes sense, because the tension in the string is huge compared to the small mass M. We can even be quantitative about this. If we tilt the picture sideways, we essentially have a pendulum of mass M and length l. And the 'effective gravity' force acting on the mass is $2mg = (2mg/M)M$. So for all the mass M knows, it is in a world where gravity has strength $g' = 2mg/M$. The frequency of such a pendulum is $\sqrt{g'/l} = \sqrt{(g/l)(2m/M)}$. ♣

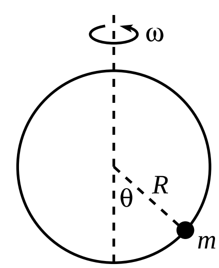

Figure 5.38

14. **Bead on rotating hoop**

Let θ be the angle the radius to the bead makes with the vertical (see Fig. 5.38). Breaking the velocity up into the part along the hoop plus the part perpendicular to the hoop, we find

$$L = \frac{1}{2}m(\omega^2R^2\sin^2\theta + R^2\dot\theta^2) + mgR\cos\theta. \quad (5.163)$$

The equation of motion is

$$R\ddot{\theta} = \sin\theta(\omega^2 R\cos\theta - g). \tag{5.164}$$

This just says that the component of gravity pulling downward along the hoop accounts for the acceleration along the hoop plus the component of the centripetal acceleration along the hoop.

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$. The right-hand side of eq. (5.164) equals 0 when either $\sin\theta = 0$ (i.e., $\theta = 0$ or $\theta = \pi$) or $\cos\theta = g/(R\omega^2)$. Since $\cos\theta$ must be less than 1, this second condition is possible only when $\omega^2 > g/R$. So we have two cases

- If $\omega^2 < g/R$, then $\theta = 0$ and $\theta = \pi$ are the only equilibrium points.

  The $\theta = \pi$ case is of course unstable. This can be seen mathematically by letting $\theta \equiv \pi + \delta$, where δ is small. Eq. (5.164) then becomes

  $$\ddot{\delta} - \delta(g/R + \omega^2) = 0. \tag{5.165}$$

  The coefficient of δ is negative, so this does not admit oscillatory solutions.

  The $\theta = 0$ case turns out to be stable. For small θ, eq. (5.164) becomes

  $$\ddot{\theta} + \theta(g/R - \omega^2) = 0. \tag{5.166}$$

  The coefficient of θ is positive, so we have sinusoidal solutions. The frequency of small oscillations is $\sqrt{g/R - \omega^2}$. This goes to 0 as $\omega \to \sqrt{g/R}$.

- If $\omega^2 > g/R$, then $\theta = 0$, $\theta = \pi$, and $\cos\theta_0 \equiv g/(R\omega^2)$ are all equilibrium points. But $\theta = 0$ is unstable because the coefficient of θ in eq. (5.166) is negative. (Similarly for $\theta = \pi$).

  So $\cos\theta_0 \equiv g/(R\omega^2)$ is the only stable equilibrium. To find the frequency of small oscillations, let $\theta \equiv \theta_0 + \delta$ in eq. (5.164), and expand to first order in δ. Using $\cos\theta_0 \equiv g/(R\omega^2)$, we find

  $$\ddot{\delta} + \omega^2 \sin^2\theta_0 \delta = 0. \tag{5.167}$$

  Therefore, the frequency of small oscillations is $\omega\sin\theta_0 = \sqrt{\omega^2 - g^2/R^2\omega^2}$.

  REMARK: This frequency goes to 0 as $\omega \to \sqrt{g/R}$. And it goes to ∞ as $\omega \to \infty$. This second limit can be looked at in the following way. For very large ω, gravity is not very important, and the bead essentially feels a centrifugal force of $mR\omega^2$ as it moves near $\theta = \pi/2$. So for all the bead knows, it is a pendulum of length R in a world where 'gravity' pulls sideways with a force $mR\omega^2 \equiv mg'$. The frequency of such a pendulum is $\sqrt{g'/R} = \sqrt{R\omega^2/R} = \omega$. ♣

The frequency $\omega = \sqrt{g/R}$ is the critical frequency above which there is a stable equilibrium at $\theta \neq 0$, i.e., above which the mass will want to move away from the bottom of the hoop.

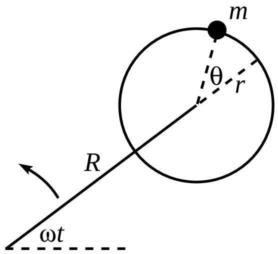

Figure 5.39

15. **Another bead on rotating hoop**

Let the angles ωt and θ be defined as in Fig. 5.39. Then the cartesian coordinates for the bead are

$$(x, y) = \Big(R\cos\omega t + r\cos(\omega t + \theta), R\sin\omega t + r\sin(\omega t + \theta)\Big). \tag{5.168}$$

The square of the speed is therefore

$$\begin{aligned} v^2 &= R^2\omega^2 + r^2(\omega+\dot\theta)^2 \\ &\quad +2Rr\omega(\omega+\dot\theta)\Big(\cos\omega t\cos(\omega t+\theta)+\sin\omega t\sin(\omega t+\theta)\Big) \\ &= R^2\omega^2 + r^2(\omega+\dot\theta)^2 + 2Rr\omega(\omega+\dot\theta)\cos\theta \end{aligned} \tag{5.169}$$

There is no potential energy, so the Lagrangian is simply $L = mv^2/2$. The equation of motion is then

$$r\ddot\theta + R\omega^2\sin\theta = 0. \tag{5.170}$$

Equilibrium occurs when $\dot\theta = \ddot\theta = 0$, and therefore when $\theta = 0$. (Well, $\theta = \pi$ also works, but that's an unstable equilibrium.) A small-angle approximation gives $\ddot\theta + (R/r)\omega^2\theta = 0$, so the frequency of small oscillations is $\Omega = \omega\sqrt{R/r}$.

REMARKS: If $R \ll r$, then $\Omega \approx 0$. This makes sense, since the frictionless hoop is essentially not moving. If $R = r$, then $\Omega = \omega$. If $R \gg r$, then Ω is very large. In this case, we can double-check the value $\Omega = \omega\sqrt{R/r}$ in the following way. In the frame of the hoop, the bead feels a centrifugal force of $m(R+r)\omega^2$. For all the bead knows, it is in a gravitational field with strength $g' \equiv (R+r)\omega^2$. So the bead (which acts like a pendulum of length r), oscillates with frequency

$$\sqrt{\frac{g'}{r}} = \sqrt{\frac{R+r}{r}\omega^2} \approx \omega\sqrt{\frac{R}{r}}, \tag{5.171}$$

for $R \gg r$.

Note that if one tries to use this 'effective gravity' argument as a double check for smaller R values, one gets the wrong answer. For example, if $R = r$, we have would obtain an oscillation frequency of $\omega\sqrt{2R/r}$ instead of the correct value $\omega\sqrt{R/r}$. This is because in reality the centrifugal force fans out near the equilibrium point, while our 'effective' gravity' argument assumes that the field lines are parallel (and so it gives a frequency that is too large). ♣

16. **Rotating curve**

In terms of the variable x, the speed along the curve is $\dot x\sqrt{1+f'^2}$, and the speed perpendicular to the curve is ωx. So the Lagrangian is

$$\frac{1}{2}m\Big(\omega^2x^2 + \dot x^2(1+f'^2)\Big) - mgf(x). \tag{5.172}$$

The equation of motion is

$$\ddot x(1+f'^2) = \dot x^2 f'f'' + \omega^2 x - gf'. \tag{5.173}$$

Equilibrium occurs when $\dot x = \ddot x = 0$, and therefore at the x_0 for which

$$x_0 = \frac{gf'(x_0)}{\omega^2}. \tag{5.174}$$

(This simply says that the component of gravity along the curve accounts for the component of the centripetal acceleration along the curve.) Using our explicit form $f = b(x/a)^\lambda$, we find

$$x_0 = a\left(\frac{\omega^2a^2}{\lambda gb}\right)^{1/(\lambda-2)}. \tag{5.175}$$

As $\lambda \to \infty$, x_0 goes to a, as it should, since the curve essentially equals zero up to a, whereupon it rises very steeply. The reader can check numerous other limits.

Letting $x \equiv x_0 + \delta$ in eq. (5.173), and expanding to first order in δ, gives

$$\ddot{\delta}\Big(1 + f'(x_0)^2\Big) = \delta\Big(\omega^2 - gf''(x_0)\Big). \tag{5.176}$$

So the frequency of small oscillations is

$$\Omega^2 = \frac{gf''(x_0) - \omega^2}{1 + f'(x_0)^2}. \tag{5.177}$$

Using the explicit form of f, along with eq. (5.175), gives

$$\Omega^2 = \frac{(\lambda - 2)\omega^2}{1 + \frac{a^2\omega^4}{g^2}\left(\frac{a^2\omega^2}{\lambda g b}\right)^{2/(\lambda-2)}}. \tag{5.178}$$

We see that λ must be greater than 2 in order for there to be oscillatory behavior around the equilibrium point. For $\lambda < 2$, the equilibrium point is unstable, i.e., to the left the force is inward, and to the right the force is outward.

In the case $\lambda = 2$, the equilibrium condition, eq. (5.174), gives $x_0 = (2gb/a^2\omega^2)x_0$. For this to be true for some x_0, we must have $\omega^2 = 2gb/a^2$. But if this holds, then eq. (5.174) is true for all x. So in this special case, the particle feels no tangential force anywhere along the curve. (In the frame of the curve, the tangential components of the centrifugal and gravitational forces exactly cancel at all points.) If $\omega^2 \neq 2gb/a^2$, then the particle feels a force either always inward or always outward.

REMARKS: For $\omega \to 0$, we have $x_0 \to 0$ and $\Omega \to 0$. And for $\omega \to \infty$, we have $x_0 \to \infty$ and $\Omega \to 0$. In both cases $\Omega \to 0$, because in both case the equilibrium position is at a place where the curve is very flat (horizontally or vertically, respectively), so the restoring force is very small.

For $\lambda \to \infty$, we have $x_0 \to a$ and $\Omega \to \infty$. The frequency is large here because the equilibrium position at a is where the curve has an abrupt corner, so the restoring force changes quickly with position. Or, you can think of it as a pendulum with a very small length (if you approximate the 'corner' by a tiny circle). ♣

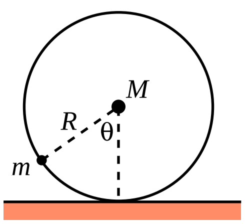

Figure 5.40

17. **Mass on wheel**

Let the angle θ be defined as in Fig. 5.40 (with the convention that θ is positive if M is to the right of m). Then the position of m in cartesian coordinates, relative to the point where m would be in contact with the ground, is

$$(x, y)_m = R(\theta - \sin\theta, 1 - \cos\theta). \tag{5.179}$$

The square of the speed of m is therefore $v_m^2 = 2R^2\dot{\theta}^2(1 - \cos\theta)$.

The position of M is $(x, y)_M = R(\theta, 1)$, so the square of its speed is $v_M^2 = R^2\dot{\theta}^2$. The Lagrangian is therefore

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1 - \cos\theta) - mgR(1 - \cos\theta). \tag{5.180}$$

The equation of motion is

$$MR^2\ddot{\theta} + 2mR^2\ddot{\theta}(1 - \cos\theta) + mR^2\dot{\theta}^2\sin\theta + mgR\sin\theta = 0. \tag{5.181}$$

In the case of small oscillations, we may use $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$. The second and third terms above are third order in θ and may be neglected. So we find

$$\ddot{\theta} + \frac{mg}{MR}\theta = 0. \tag{5.182}$$

The frequency of small oscillations is

$$\omega = \sqrt{\frac{m}{M}}\sqrt{\frac{g}{R}}. \tag{5.183}$$

REMARKS: If $M \gg m$, then $\omega \to 0$. This makes sense.

If $m \gg M$, then $\omega \to \infty$. This also makes sense, because the huge mg force makes the situation similar to one where the wheel is bolted to the floor, in which case the wheel vibrates with a high frequency.

Eq. (5.183) can actually be written down without doing any calculations. We'll let the reader show that for small oscillations the gravitational force on m has the effect of essentially applying a sideways force on M equal to $-mg\theta$. So the horizontal $F = Ma$ equation for M is $MR\ddot{\theta} = -mg\theta$, from which the result follows. ♣

18. **Double pendulum**

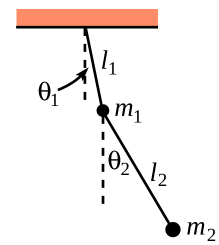

Figure 5.41

Relative to the pivot point, the cartesian coordinates of m_1 and m_2 are, respectively (see Fig. 5.41),

$$\begin{aligned} (x,y)_1 &= (\ell_1 \sin\theta_1, -\ell_1\cos\theta_1), \\ (x,y)_2 &= (\ell_1 \sin\theta_1 + \ell_2\sin\theta_2, -\ell_1\cos\theta_1 - \ell_2\cos\theta_2). \end{aligned} \tag{5.184}$$

The squares of the speeds are therefore

$$\begin{aligned} v_1^2 &= \ell_1^2\dot{\theta}_1^2, \\ v_2^2 &= \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2). \end{aligned} \tag{5.185}$$

The Lagrangian is then

$$\begin{aligned} L &= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\Big) \\ &\quad + m_1 g\ell_1\cos\theta_1 + m_2 g(\ell_1\cos\theta_1 + \ell_2\cos\theta_2). \end{aligned} \tag{5.186}$$

The equations of motion from varying θ_1 and θ_2 are

$$\begin{aligned} 0 &= (m_1+m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) \\ &\quad + (m_1+m_2)g\ell_1\sin\theta_1, \\ 0 &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) \\ &\quad + m_2 g\ell_2\sin\theta_2. \end{aligned} \tag{5.187}$$

This is a rather large mess, but it simplifies greatly if we consider small oscillations. Using the small-angle approximations and keeping only the leading-order terms, we have

$$\begin{aligned} 0 &= (m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1, \\ 0 &= \ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1 + g\theta_2. \end{aligned} \tag{5.188}$$

- Consider the special case $\ell_1 = \ell_2 \equiv \ell$. We may find the frequencies of the normal modes using the usual determinant method. They are

$$\omega_\pm = \sqrt{\frac{m_1 + m_2 \pm \sqrt{m_1 m_2 + m_2^2}}{m_1}}\sqrt{\frac{g}{\ell}}. \tag{5.189}$$

The normal modes are found to be, after some simplification,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp\sqrt{m_2} \\ \sqrt{m_1+m_2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.190}$$

Some special cases are:

Case 1: $m_1 = m_2$**.** The frequencies are

$$\omega_{\pm} = \sqrt{2 \pm \sqrt{2}}\sqrt{\frac{g}{\ell}}. \tag{5.191}$$

The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.192}$$

Case 2: $m_1 \gg m_2$**.** With $m_2/m_1 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_{\pm} = (1 \pm \sqrt{\epsilon})\sqrt{\frac{g}{\ell}}. \tag{5.193}$$

The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp\sqrt{\epsilon} \\ 1 \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.194}$$

In both modes, the upper (heavy) mass essentially stands still, and the lower (light) mass oscillates like a pendulum of length ℓ.

Case 3: $m_1 \ll m_2$**.** With $m_1/m_2 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_{+} = \sqrt{\frac{2g}{\epsilon\ell}}, \qquad \omega_{-} = \sqrt{\frac{g}{2\ell}}. \tag{5.195}$$

The normal modes are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \mp 1 \\ 1 \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.196}$$

In the first mode, the lower (heavy) mass essentially stands still, and the upper (light) mass vibrates back and forth at a high frequency (because there is a very large tension in the rods). In the second mode, the rods form a straight line, and the system is essentially a pendulum of length 2ℓ.

- Consider the special case $m_1 = m_2$. Using the determinant method, the frequencies of the normal modes are found to be

$$\omega_{\pm} = \sqrt{g}\sqrt{\frac{\ell_1 + \ell_2 \pm \sqrt{\ell_1^2 + \ell_2^2}}{\ell_1 \ell_2}}. \tag{5.197}$$

The normal modes are found to be, after some simplification,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_{\pm} = \begin{pmatrix} \ell_2 \\ \ell_2 - \ell_1 \mp \sqrt{\ell_1^2 + \ell_2^2} \end{pmatrix} \cos(\omega_{\pm} t + \phi_{\pm}). \tag{5.198}$$

Some special cases are:

Case 1: $\ell_1 = \ell_2$. This was done above.

Case 2: $\ell_1 \gg \ell_2$. With $\ell_2/\ell_1 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_+ = \sqrt{\frac{2g}{\ell_2}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_1}}. \tag{5.199}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} -\epsilon \\ 2 \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.200}$$

In the first mode, the masses essentially move equal distances in opposite directions, at a very high frequency (because ℓ_2 is so small). In the second mode, the string stays straight, and the masses move just like a mass of $2m$. The system is essentially a pendulum of length ℓ.

Case 3: $\ell_1 \ll \ell_2$. With $\ell_1/\ell_2 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_+ = \sqrt{\frac{2g}{\ell_1}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_2}}. \tag{5.201}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.202}$$

In the first mode, the bottom mass essentially stays still, while the top one oscillates at a very high frequency (because ℓ_1 is so small). The factor of 2 is in the frequency because the top mass essentially lives in a world where the acceleration from gravity is $g' = 2g$ (because of the extra mg force downward from the lower mass). In the second mode, the system is essentially a pendulum of length ℓ_2. The string is slightly bent, just enough to make the tangential force on the top mass roughly 0 (because otherwise it would oscillate at a high frequency, since ℓ_1 is so small).

19. **Pendulum with free support**

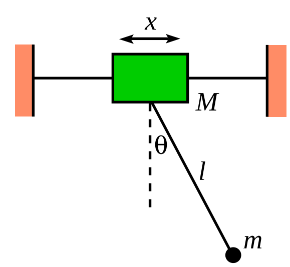

Figure 5.42

Let x be the coordinate of M. Let θ be the angle of the pendulum (see Fig. 5.42). Then the position of the mass m, in cartesian coordinates, is $(x+\ell\sin\theta, -\ell\cos\theta)$. The square of the speed of m is found to be $v_m^2 = \dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.203}$$

The equations of motion from varying x and θ are

$$\begin{aligned} (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= 0, \\ \ell\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta &= 0. \end{aligned} \tag{5.204}$$

In the event that θ is very small, we may use the small angle approximations. Keeping only terms that are first-order in θ, we obtain

$$\begin{aligned}(M+m)\ddot{x}+m\ell\ddot{\theta} &= 0,\\ \ddot{x}+\ell\ddot{\theta}+g\theta &= 0.\end{aligned} \tag{5.205}$$

The first equation is momentum conservation. Integrating it twice gives

$$x=-\frac{m\ell}{M+m}\theta+At+B. \tag{5.206}$$

The second equation is $F=ma$ in the tangential direction.
Eliminating $\ddot{x}$ from eqs. (5.205) gives

$$\ddot{\theta}+\left(\frac{M+m}{M}\right)\frac{g}{\ell}\theta=0. \tag{5.207}$$

The solution to this equation is $\theta(t)=C\cos(\omega t+\phi)$, where

$$\omega=\sqrt{1+\frac{m}{M}}\sqrt{\frac{g}{\ell}}. \tag{5.208}$$

So the general solution for x and θ is

$$\theta(t)=C\cos(\omega t+\phi), \qquad x(t)=-\frac{Cm\ell}{M+m}\cos(\omega t+\phi)+At+B. \tag{5.209}$$

The constant B is irrelevant, so let's forget it.
The two normal modes are the following.

- $A=0$: In this case $x=-\theta m\ell/(M+m)$. So the masses oscillate with frequency ω, always moving in opposite directions.
- $C=0$: In this case, $\theta=0$ and $x=At$. So both masses move horizontally with the same speed.

REMARKS: If $M\gg m$, then $\omega=\sqrt{g/\ell}$, as it should be, since the support essentially stays still.
If $m\gg M$, then $\omega\to\sqrt{m/M}\sqrt{g/\ell}\to\infty$. This makes sense, since the tension in the rod is so large. We can actually be quantitative about this limit. We'll let you show that for small oscillations and $m\gg M$, the gravitational force on m has the effect of essentially applying a sideways force on M equal to $mg\theta$. So the horizontal $F=Ma$ equation for M is $M\ddot{x}=mg\theta$, But $x\approx-\ell\theta$ in this limit, so we have $-M\ell\ddot{\theta}x=mg\theta$, from which the result follows. ♣

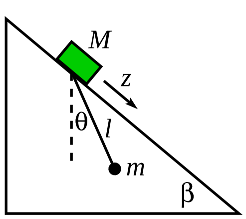

Figure 5.43

20. **Pendulum support on inclined plane**

Let z be the coordinate of M along the plane. Let θ be the angle of the pendulum (see Fig. 5.43). In cartesian coordinates, the positions of M and m are then

$$\begin{aligned}(x,y)_M &= (z\cos\beta,-z\sin\beta),\\ (x,y)_m &= (z\cos\beta+\ell\sin\theta,-z\sin\beta-\ell\cos\theta).\end{aligned} \tag{5.210}$$

The squares of the speeds are

$$\begin{aligned}v_M^2 &= \dot{z}^2,\\ v_m^2 &= \dot{z}^2+\ell^2\dot{\theta}^2+2\ell\dot{z}\dot{\theta}(\cos\beta\cos\theta-\sin\beta\sin\theta).\end{aligned} \tag{5.211}$$

The Lagrangian is therefore

$$\frac{1}{2}M\dot{z}^2+\frac{1}{2}m\Big(\dot{z}^2+\ell^2\dot{\theta}^2+2\ell\dot{z}\dot{\theta}\cos(\theta+\beta)\Big)+Mgz\sin\beta+mg(z\sin\beta+\ell\cos\theta). \quad (5.212)$$

The equations of motion from varying z and θ are

$$\begin{aligned}(M+m)\ddot{z}+m\ell\Big(\ddot{\theta}\cos(\theta+\beta)-\dot{\theta}^2\sin(\theta+\beta)\Big) &= (M+m)g\sin\beta,\\ \ell\ddot{\theta}+\ddot{z}\cos(\theta+\beta) &= -g\sin\theta. \end{aligned} \quad (5.213)$$

Let us now consider small oscillations about the equilibrium point (where $\ddot{\theta}=\dot{\theta}=0$). We first have to find where this point is. The first equation above gives $\ddot{z}=g\sin\beta$. The second equation then gives $g\sin\beta\cos(\theta+\beta)=-g\sin\theta$. By expanding this cosine, or just by inspection, we have $\theta=-\beta$ (or $\theta=\pi-\beta$, but this is an unstable equilibrium). So the equilibrium position of the pendulum is where the string is perpendicular to the plane. (This makes sense. Because the tension in the string is orthogonal to the plane, for all the pendulum bob knows, it may as well simply be sliding down a plane parallel to the given one, a distance ℓ away. Given the same initial speed, the two masses will slide down their two 'planes' at the same speed at all times.)

To find the normal modes and frequencies of small oscillations, let $\theta\equiv-\beta+\delta$, and expand eqs. (5.213) to first order in δ. Letting $\ddot{\eta}\equiv\ddot{z}-g\sin\beta$ for convenience, we have

$$\begin{aligned}(M+m)\ddot{\eta}+m\ell\ddot{\delta} &= 0,\\ \ddot{\eta}+\ell\ddot{\delta}+\delta g\cos\beta &= 0.\end{aligned} \quad (5.214)$$

Using the determinant method, the frequencies of the normal modes are found to be

$$\omega=0, \qquad \omega=\sqrt{1+\frac{m}{M}}\sqrt{\frac{g\cos\beta}{\ell}}. \quad (5.215)$$

This is the same answer as in the previous problem (with a horizontal plane), but with $g\cos\beta$ instead of g. This makes sense, because in a frame accelerating down the plane at $g\sin\beta$, the only external force on the masses is an effective gravity force $g\cos\beta$ perpendicular to the plane. For all M and m know, they live in a world where gravity has strength $g'=g\cos\beta$.

21. **Tilting plane**

Relative to the support, the position of M is $(\ell\sin\theta,-\ell\cos\theta)$. The position of m is $(\ell\sin\theta+x\cos\theta,-\ell\cos\theta+x\sin\theta)$. The squares of the velocities are therefore

$$v_M^2=\ell^2\dot{\theta}^2, \qquad v_m^2=(\ell\dot{\theta}+\dot{x})^2+x^2\dot{\theta}^2. \quad (5.216)$$

(v_m^2 can also be obtained without taking the derivative of the position; $(\ell\dot{\theta}+\dot{x})$ is the speed along the long rod, and $x\dot{\theta}$ is the speed perpendicular to it.) The Lagrangian is

$$L=\frac{1}{2}M\ell^2\dot{\theta}^2+\frac{1}{2}m\Big((\ell\dot{\theta}+\dot{x})^2+x^2\dot{\theta}^2\Big)+Mg\ell\cos\theta+mg(\ell\cos\theta-x\sin\theta). \quad (5.217)$$

The equations of motion from varying x and θ are

$$\begin{aligned}(\ell\ddot{\theta}+\ddot{x}) &= x\dot{\theta}^2-g\sin\theta,\\ M\ell^2\ddot{\theta}+m\ell(\ell\ddot{\theta}+\ddot{x})+mx^2\ddot{\theta}+2mx\dot{x}\dot{\theta} &= -Mg\ell\sin\theta-mg\ell\sin\theta\\ &\quad -mgx\cos\theta.\end{aligned} \quad (5.218)$$

Let us now consider the case where both x and θ are small (or more precisely, $\theta \ll 1$ and $x/\ell \ll 1$). Expanding eqs. (5.218) to first order in θ and x/ℓ gives

$$\begin{aligned} (\ell\ddot{\theta} + \ddot{x}) + g\theta &= 0, \\ M\ell(\ell\ddot{\theta} + g\theta) + m\ell(\ell\ddot{\theta} + \ddot{x}) + mg\ell\theta + mgx &= 0. \end{aligned} \tag{5.219}$$

We can simplify these a bit. Using the first equation to substitute $-g\theta$ for $(\ell\ddot{\theta} + \ddot{x})$ and also $-\ddot{x}$ for $(\ell\ddot{\theta} + g\theta)$ in the second equation gives

$$\begin{aligned} \ell\ddot{\theta} + \ddot{x} + g\theta &= 0, \\ -M\ell\ddot{x} + mgx &= 0. \end{aligned} \tag{5.220}$$

The normal modes can be found using the determinant method, or we can find them just by inspection. The second equation says that either $x(t) \equiv 0$ or $x(t) = A\cosh(\alpha t + \beta)$, where $\alpha = \sqrt{mg/M\ell}$. So we have two cases.

- If $x(t) = 0$, then the first equation in (5.220) says that the normal mode is

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = C \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(\omega t + \phi), \tag{5.221}$$

  where $\omega \equiv \sqrt{g/\ell}$.
  This mode is fairly obvious. With proper initial conditions, m will stay right where M is. The normal force from the long rod will be exactly what is needed in order for m to undergo the same oscillatory motion as M.

- If $x(t) = A\cosh(\alpha t + \beta)$, then the first equation in (5.220) can be solved to give the normal mode

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = C \begin{pmatrix} -m \\ \ell(M + m) \end{pmatrix} \cosh(\alpha t + \beta), \tag{5.222}$$

  where $\alpha = \sqrt{mg/M\ell}$.
  This mode is not so obvious. And indeed, its range of validity is rather limited. The exponential behavior will quickly make x and θ become large, and thus outside the validity of our small-variable approximations. Note that in this mode the center-of-mass remains fixed.

22. **Motion on a cone**

Let the particle be a distance r from the axis. Then its height is $r/\tan\alpha$, and the distance up along the cone is $r/\sin\alpha$. Let θ be the angle around the cone. Breaking the velocity into the components up along the cone and around the cone, we see that the square of the speed is $v^2 = \dot{r}^2/\sin^2\alpha + r^2\dot{\theta}^2$. The Lagrangian is therefore

$$L = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}. \tag{5.223}$$

The equations of motion from varying θ and r are

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0 \\ \ddot{r} &= r\dot{\theta}^2\sin^2\alpha - g\cos\alpha\sin\alpha. \end{aligned} \tag{5.224}$$

(The first of these is conservation of angular momentum. The second one is more transparent if we divide through by $\sin\alpha$. With $x \equiv r/\sin\alpha$ being the distance up along the cone, it becomes $\ddot{x} = (r\dot{\theta}^2)\sin\alpha - g\cos\alpha$. This is just $F = ma$ in the x direction.)

Letting $mr^2\dot{\theta} \equiv L$, we may eliminate $\dot{\theta}$ from the second equation to obtain

$$\ddot{r} = \frac{L^2 \sin^2\alpha}{m^2 r^3} - g\cos\alpha\sin\alpha. \tag{5.225}$$

Let us now calculate the two desired frequencies.

- Frequency of circular oscillations, ω:
  For circular motion with $r = r_0$, we have $\dot{r} = \ddot{r} = 0$, so eq. (5.224) gives

$$\omega \equiv \dot{\theta} = \sqrt{\frac{g}{r_0 \tan\alpha}}. \tag{5.226}$$

- Frequency of oscillations about a circle, Ω:
  If the orbit were actually the circle $r = r_0$, then eq. (5.225) would give (with $\ddot{r} = 0$)

$$\frac{L^2 \sin^2\alpha}{m^2 r_0^3} = g\cos\alpha\sin\alpha. \tag{5.227}$$

  (Of course, writing L as $mr_0^2\dot{\theta}$, this is equivalent to eq. (5.226).)
  We will now use our standard procedure of letting $r(t) = r_0 + \delta(t)$, where $\delta(t)$ is very small, and then plugging this into eq. (5.225) and expanding to first order in δ. Using

$$\frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.228}$$

  we have

$$\ddot{\delta} = \frac{L^2\sin^2\alpha}{m^2 r_0^3}\left(1 - \frac{3\delta}{r_0}\right) - g\cos\alpha\sin\alpha. \tag{5.229}$$

  Recalling eq. (5.227), we obtain a bit of cancellation and are left with

$$\ddot{\delta} = -\frac{3\delta L^2 \sin^2\alpha}{m^2 r_0^4} \tag{5.230}$$

  Using eq. (5.227) again to eliminate L (technically, eq. (5.227) only holds for circular motion, but any error is of higher order in the following equation), we have

$$\ddot{\delta} + \delta\frac{3g}{r_0}\sin\alpha\cos\alpha = 0. \tag{5.231}$$

  So we find

$$\Omega = \sqrt{\frac{3g}{r_0}\sin\alpha\cos\alpha} = (\sqrt{3}\sin\alpha)\omega. \tag{5.232}$$

  Thus, the ratio Ω/ω is independent of r_0.

The two frequencies are equal if $\sin\alpha = 1/\sqrt{3}$, i.e., $\alpha \approx 35.3^\circ \equiv \tilde{\alpha}$. If $\alpha = \tilde{\alpha}$, then after one revolution r returns to the value it had at the beginning of the revolution. So the particle undergoes periodic motion.

If $\alpha > \tilde{\alpha}$, then $\Omega > \omega$ (i.e., the r value goes through a whole oscillation before one complete circle is traversed). If $\alpha < \tilde{\alpha}$, then $\Omega < \omega$ (i.e., more than one complete circle is needed for the r value to go through one cycle.)

REMARKS: In the limit where $\alpha \to 0$, eq. (5.232) says that $\Omega/\omega \to 0$. (In fact, eqs. (5.226) and (5.232) say that $\omega \to \infty$ and $\Omega \to 0$). So the particle spirals around many times during one complete r cycle. (This seems intuitive.)

In the limit where $\alpha \to \pi/2$ (i.e., the cone is almost a flat plane) eq. (5.232) says that $\Omega/\omega \to \sqrt{3}$. (Both Ω and ω go to 0). This result is not at all obvious (at least to me).

If $\Omega/\omega = \sqrt{3}\sin\alpha$ is a rational number, then the particle will undergo periodic motion. For example, if $\alpha = 60°$, then $\Omega/\omega = 3/2$; it takes two complete circles for r to go through three cycles. Or, if $\alpha = \arcsin(1/2\sqrt{3}) \approx 16.8°$, then $\Omega/\omega = 1/2$; it takes two complete circles for r to go through one cycle; etc.

We calculated Ω above by letting $r = r_0 + \delta$ and then expanding, but one could also use the effective potential method. Eq. (5.225) yields an effective potential

$$V_{\text{eff}} = \frac{L^2 \sin^2\alpha}{2mr^2} + mgr\cos\alpha\sin\alpha. \tag{5.233}$$

The frequency of small oscillations about the minimum at r_0 is then

$$\Omega = \sqrt{\frac{V''_{\text{eff}}(r_0)}{m}}. \tag{5.234}$$

The quantity L may be eliminated in favor of r_0 by using the condition $V'_{\text{eff}}(r_0) = 0$, and we'll leave it to the reader to show that this reproduces eq. (5.232). ♣

23. **Shortest distance in a plane**

Let the two given points be (x_1, y_1) and (x_2, y_2) Let the path be described by the function $y(x)$. (Yes, we'll assume it can be written as a function. Locally, we don't have to worry about any double-valued issues.) Then the length of the path is

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx. \tag{5.235}$$

The 'Lagrangian' is $L = \sqrt{1 + y'^2}$, and the Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dx}\frac{\partial L}{\partial y'} &= \frac{\partial L}{\partial y} \\ \Longrightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) &= 0. \end{aligned} \tag{5.236}$$

So $y'/\sqrt{1+y'^2}$ is constant. Therefore, y' is also constant, and we have the straight line $y(x) = Ax + B$, where A and B are determined from the endpoint conditions.

24. **Minimal surface**

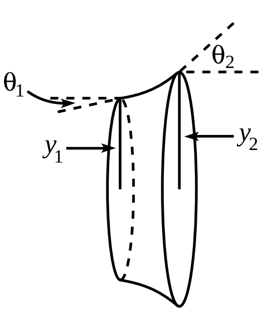

Figure 5.44

The tension throughout the surface is constant, since it is in equilibrium. The ratio of the circumferences of the circular boundaries of the ring is y_2/y_1. The condition that the horizontal forces on the ring cancel is therefore $y_1\cos\theta_1 = y_2\cos\theta_2$, where the θ's are the angles of the surface, as shown in Fig. 5.44. In other words, $y\cos\theta$ is constant throughout the surface. But $\cos\theta = 1/\sqrt{1+y'^2}$, so we have

$$\frac{y}{\sqrt{1+y'^2}} = C. \tag{5.237}$$

This is the same as eq. (5.73), and the solution proceeds as in Section 5.8.

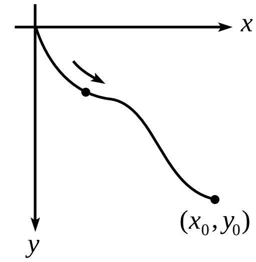

Figure 5.45

25. **The Brachistochrone**

(a) In Fig. 5.45, the boundary conditions are $y(0) = 0$ and $y(x_0) = y_0$ (with downward taken to be the positive y direction). From energy conservation, the speed at position y is $v = \sqrt{2gy}$. The total time is therefore

$$T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx. \tag{5.238}$$

Our 'Lagrangian' is thus

$$L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}. \tag{5.239}$$

We can apply the E-L equation now, or we can simply use Lemma 5.5, with $f(y) = 1/\sqrt{y}$. Eg. (5.84) gives

$$fy'' = f'(1+y'^2) \quad \Longrightarrow \quad \frac{y''}{\sqrt{y}} = -\frac{1+y'^2}{2y\sqrt{y}} \quad \Longrightarrow \quad -2yy'' = 1+y'^2, \tag{5.240}$$

as desired. And eq. (5.86) gives

$$1+y'^2 = Cf(y)^2 \quad \Longrightarrow \quad 1+y'^2 = \frac{C}{y}, \tag{5.241}$$

as desired.

(b) At this point we can either simply verify that the cycloid solution satisfies eq. (5.241), or we can separate variables and then integrate eq. (5.241). The latter method has the advantage, of course, of not requiring the solution to be given. Let's do it both ways.

Verification: Assume $x = a(\theta - \sin\theta)$, and $y = a(1-\cos\theta)$. Then

$$y' \equiv \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}. \tag{5.242}$$

Therefore,

$$1+y'^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta} = \frac{2a}{y}, \tag{5.243}$$

which agrees with eq. (5.241), with $C \equiv 2a$.

Separation of variables: Solving for y' in eq. (5.241) and separating variables gives

$$\frac{\sqrt{y}\,dy}{\sqrt{C-y}} = \pm dx. \tag{5.244}$$

A helpful change of variables to get rid of the square root in the denominator is $y \equiv C\sin^2\phi$. Then $dy = 2C\sin\phi\cos\phi\,d\phi$, and eq. (5.244) simplifies to

$$2C\sin^2\phi d\phi = \pm dx. \tag{5.245}$$

Integrating this (using $\sin^2\phi = (1-\cos 2\phi)/2$) gives $C(2\phi - \sin 2\phi) = \pm 2x$ (plus an irrelevant constant).

Note that we may rewrite our definition of ϕ (which was $y \equiv C\sin^2\phi$) as $2y = C(1-\cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have

$$x = \pm a(\theta - \sin\theta), \qquad y = a(1-\cos\theta). \tag{5.246}$$

where $a \equiv C/2$.

26. **Index of refraction**

Let the path be described by $y(x)$. The speed at height y is $v \propto y$. The time to go from (x_0, y_0) to (x_1, y_1) is therefore

$$T = \int_{x_0}^{x_1} \frac{ds}{v} \propto \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx. \tag{5.247}$$

Our 'Lagrangian' is thus

$$L \propto \frac{\sqrt{1+y'^2}}{y}. \tag{5.248}$$

We can apply the E-L equation now, or we can simply use Lemma 5.5, with $f(y) = 1/y$. Eg. (5.84) gives

$$fy'' = f'(1+y'^2) \quad\Longrightarrow\quad \frac{y''}{y} = -\frac{1+y'^2}{y^2} \quad\Longrightarrow\quad -yy'' = 1+y'^2. \tag{5.249}$$

And eq. (5.86) gives

$$1+y'^2 = Bf(y)^2 \quad\Longrightarrow\quad 1+y'^2 = \frac{B}{y^2}. \tag{5.250}$$

We must now integrate this. Solving for y', and then separating variables and integrating, gives

$$\int dx = \pm \int \frac{y\,dy}{\sqrt{B-y^2}} \quad\Longrightarrow\quad x + A = \mp\sqrt{B-y^2}\,. \tag{5.251}$$

Hence, $(x+A)^2 + y^2 = B$, which is the equation for a circle. Note that the circle is centered at a point on the bottom of the slab. (This must be the point where the perpendicular bisector of the line joining the two given points intersects the bottom of the slab.)

27. **Existence of minimal surface**

From Section 5.8, the general solution takes the form

$$y(x) = \frac{1}{b}\cosh b(x+d). \tag{5.252}$$

If we choose the origin to be midway between the rings, we have $d = 0$. Both boundary condition are then

$$c = \frac{1}{b}\cosh ba. \tag{5.253}$$

If a/c is larger than a certain critical value, then there is no solution for b, as we now show.

Let us define the dimensionless quantities,

$$\eta \equiv \frac{a}{c}, \qquad \text{and} \qquad z \equiv bc. \tag{5.254}$$

Then eq. (5.253) becomes

$$z = \cosh \eta z. \tag{5.255}$$

If we (roughly) plot the graphs of $w = z$ and $w = \cosh \eta z$ for a few values of η (see Fig. 5.46), we see that there is no solution for z if η is too big. The limiting value

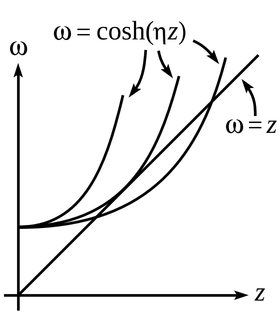

Figure 5.46

for η for which a solution exists occurs when the curves $w = z$ and $w = \cosh \eta z$ are tangent, that is, when the functions are equal and the slopes are equal. Let η_0 be this critical value for η. Let z_0 be the place where the tangency occurs. Then equality of the values and slopes gives

$$z_0 = \cosh(\eta_0 z_0), \qquad \text{and} \qquad 1 = \eta_0 \sinh(\eta_0 z_0). \tag{5.256}$$

Dividing the second of these equations by the first gives

$$1 = (\eta_0 z_0) \tanh(\eta_0 z_0). \tag{5.257}$$

This must be solved numerically. The solution is

$$\eta_0 z_0 \approx 1.200. \tag{5.258}$$

Plugging this into the second of eqs. (5.256) gives

$$\left(\frac{a}{c}\right)_{\text{max}} \equiv \eta_0 \approx 0.663. \tag{5.259}$$

If a/c is larger than 0.663, then the Euler-Lagrange equation has no solution that is consistent with the boundary conditions. (In other words, there is no surface which is stationary with respect to small perturbations.) Above this value of a/c, a soap bubble minimizes its area by heading toward the shape of just two discs, but it will pop long before it reaches this configuration.

REMARK: How does the area of the limiting minimal surface compare with the area of the two circles?
The area of the two circles is

$$A_c = 2\pi c^2. \tag{5.260}$$

The area of the critical surface is

$$A_s = \int_{-a}^{a} 2\pi y \sqrt{1 + y'^2}\, dx. \tag{5.261}$$

Using eq. (5.252), with $d = 0$, we find

$$\begin{aligned} A_s &= \int_{-a}^{a} \frac{2\pi}{b} \cosh^2 bx\, dx \\ &= \int_{-a}^{a} \frac{\pi}{b}(1 + \cosh 2bx)\, dx \\ &= \frac{2a\pi}{b} + \frac{\pi \sinh 2ba}{b^2}. \end{aligned} \tag{5.262}$$

But from the definitions of η and z, we have $a = \eta_0 c$ and $b = z_0/c$ for the critical surface. Therefore,

$$A_s = \pi c^2 \left(\frac{2\eta_0}{z_0} + \frac{\sinh 2\eta_0 z_0}{z_0^2} \right). \tag{5.263}$$

Plugging in the numerical values ($\eta_0 \approx 0.663$ and $z_0 \approx 1.810$) gives

$$A_c \approx (6.28)\pi c^2, \qquad \text{and} \qquad A_s \approx (7.54)\pi c^2. \tag{5.264}$$

(The ratio of these areas is approximately 1.2, which is actually $\eta_0 z_0$. We'll let you prove this.) The critical surface therefore has a larger area. This is expected, of course, because for $a/c > \eta_0$ the surface tries to run off to one with a smaller area, and there are no other stable configurations besides the cosh solution we found. ♣