Chapter 4

Conservation of Energy and Momentum

Conservation laws are extremely important in physics. They are an enormous help (both quantitatively and qualitatively) in figuring out what is going on in a physical system.

When we say that something is 'conserved', we mean that it is constant over time. If a certain quantity is conserved, for example, while a ball rolls around on a hill, or while a group of particles interact, then the possible final motions are greatly restricted. If we can write down enough conserved quantities (which we are generally able to do), then we can restrict the final motions down to just one possibility, and so we've solved our problem. Conservation of energy and momentum are two of the main conservation laws in physics. A third, conservation of angular momentum, is discussed in Chapters 7 and 8.

It should be noted that it is not *necessary* to use conservation of energy and momentum when solving a problem. We will derive these conservation laws from Newton's laws; therefore, if you felt like it, you could always simply start with first principles and use $F = ma$, etc. You would, however, soon grow weary of this approach. The point of conservation laws is that they make your life easier, and they provide a means for getting a good idea of the overall behavior of a given system.

4.1 Conservation of energy in 1-D

Consider a force in one-dimension that depends on only position, that is, $F = F(x)$. If we write a as $v\,dv/dx$, then $F = ma$ gives

$$mv\frac{dv}{dx} = F(x). \tag{4.1}$$

Separating variables and integrating gives $mv^2/2 = E + \int_{x_0}^{x} F(x')dx'$, where E is a constant of integration, dependent on the choice of x_0 (we already knew this from

Section 2.3). Therefore, if we define the *potential energy*, $V(x)$, as

$$V(x) \equiv -\int_{x_0}^{x} F(x')dx', \tag{4.2}$$

then we may write

$$\frac{1}{2}mv^2 + V(x) = E. \tag{4.3}$$

We define the first term here to be the kinetic energy. This equation is true at any point in the particle's motion; hence, the sum of the kinetic energy and potential energy is a constant.

The constant E depends, of course, on the arbitrary choice of x_0 in eq. (4.2). E and $V(x)$ have no meaning by themselves. Only differences in E and $V(x)$ are relevant, because these differences are independent of the choice of x_0. For example, it makes no sense to say that the gravitational potential energy of an object at height y is $-\int F\,dy = -\int(-mg)\,dy = mgy$. You have to say that it is mgy *with respect to the ground* (if your x_0 is at ground level). If you wanted to, you could say that the potential energy is $mgy + 7mg$ with respect to a point 7 meters below the ground. This is perfectly correct, albeit a bit cumbersome.[1]

If we take the difference between eq. (4.3) evaluated at two points, x_1 and x_2, then we obtain

$$\frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1) = V(x_1) - V(x_2) = \int_{x_1}^{x_2} F(x')dx'. \tag{4.4}$$

Here it is clear that only differences in energies matter. If we define the right-hand side of eq. (4.4) be the *work* done on the particle as it moves from x_1 to x_2, then we have produced the *work-energy theorem*,

Theorem 4.1 *The change in a particle's kinetic energy between points x_1 and x_2 is equal to the work done on the particle between x_1 and x_2.*

In Boston, lived Jack as did Jill,
Who gained *mgh* on a hill.
In their liquid pursuit,
Jill exclaimed with a hoot,
"Jack, I think we've just climbed a landfill!"

While noting, "Oh, this is just grand,"
Jack tripped on some trash in the sand.
He changed his potential
To kinetic, torrential,
But not before grabbing Jill's hand.

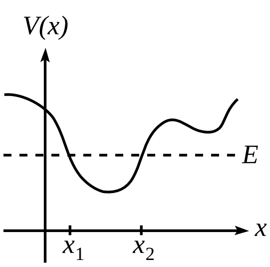


Figure 4.1

Having chosen a reference point for the energies, if we draw the $V(x)$ curve and also the constant E line (see Fig. 4.1), then the difference between them gives

[1]It gets to be a pain to keep repeating "with respect to the ground". Therefore, whenever anyone talks about gravitational potential energy in an experiment at the surface of the earth, it is understood that the ground is the reference point. If, on the other hand, the experiment reaches out to distances far from the earth, then $r = \infty$ is understood to be the reference point, for reasons of convenience we will shortly see.

the kinetic energy. The places where $V(x) > E$ are the regions where the particle cannot go. The places where $V(x) = E$ are the "turning points", where the particle stops and changes direction. In the figure, the particle is trapped between x_1 and x_2, and oscillates back and forth. The potential $V(x)$ is extremely useful this way, because it makes clear the general properties of the motion.

REMARK: It may seem silly to introduce a specific x_0 as a reference point, since it is only eq. (4.4) that has any meaning. (It's somewhat like taking the difference between 17 and 8 by first finding their sizes relative to 5, namely 12 and 3, and then subtracting 3 from 12 to obtain 9.) But it is extremely convenient to be able to label all positions with a definite number, $V(x)$, and then take differences between the V's when needed. ♣

Note that eq. (4.2) implies

$$F(x) = -\frac{dV(x)}{dx}. \tag{4.5}$$

Given $V(x)$, it is easy to take its derivative to obtain $F(x)$. But given $F(x)$, it may be difficult (or impossible) to perform the integration in eq. (4.2) and write $V(x)$ in closed form. But this is not of much concern. The function $V(x)$ is well-defined, and if needed it can be computed numerically to any desired accuracy.

It is quite obvious that F needs to be a function of only x in order for the $V(x)$ in eq. (4.2) to be a well-defined function. If F depended on t or v, then $V(x_1) - V(x_2)$ would be path-dependent. That is, it would depend on *when* or *how quickly* the particle went from x_1 to x_2. Only for $F = F(x)$ is there no ambiguity in the result. (When dealing with more than one spatial dimension, however, there may be an ambiguity in V. This is the topic of Section 4.3.)

Example (Gravitational potential energy): Consider two point masses, M and m, separated by a distance r. The (attractive) gravitational force between them is GMm/r^2 (Newton's law of gravitation). The potential energy of the system at separation r, measured relative to separation r_0, is

$$V(r) - V(r_0) = -\int_{r_0}^{r} \frac{-GMm}{r'^2} dr' = \frac{-GMm}{r} + \frac{GMm}{r_0}. \tag{4.6}$$

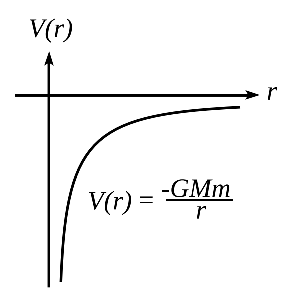


Figure 4.2

A convenient choice for r_0 is ∞, since this makes the second term vanish. It will be understood from now on that this $r_0 = \infty$ reference point has been chosen. Therefore (see Fig. 4.2),

$$V(r) = \frac{-GMm}{r}. \tag{4.7}$$

Here's a somewhat silly exercise. What is the gravitational potential energy of a mass m at height y, relative to the ground? We know, of course, that it is mgy, but let's do it the hard way. If M is the mass of the earth, and R is its radius (with $R \gg y$), then

$$V(R+y) - V(R) = \frac{-GMm}{R+y} - \frac{-GMm}{R}$$

$$
\begin{aligned}
&= \frac{-GMm}{R}\left(\frac{1}{1+y/R}-1\right)\\
&\approx \frac{-GMm}{R}\Big((1-y/R)-1\Big)\\
&= \frac{GMmy}{R^2},
\end{aligned}
\tag{4.8}
$$

where we have used the Taylor series approximation for $1/(1+\epsilon)$ to obtain the third line. (We have also used the fact that a sphere can be treated like a point mass, as far as gravity is concerned. We will prove this in Section 4.4.1.) But $g \equiv GM/R^2$, so the potential difference is mgy. We have, of course, simply gone around in circles here. We integrated in eq. (4.6), and then basically differentiated in eq. (4.8) by taking the difference between the forces. But it's good to check that everything works out.

You are encouraged to do Problem 10 at this point.

REMARK: A good way to visualize a potential $V(x)$ is to imagine a ball sliding around in a valley or on a hill. For example, the potential of a typical spring is $V(x) = kx^2/2$ (which produces the Hooke's law force, $F(x) = -dV/dx = -kx$), and we can get a decent idea of what's going on if we imagine a valley with height given by $y = x^2/2$. The gravitational potential of the ball is then $mgy = mgx^2/2$. Choosing $mg = k$ gives the desired potential. If we then look at the projection of the ball's motion onto the x-axis, it seems like we have constructed a setup identical to the original spring.

However, although this analogy helps in visualizing the basic properties of the motion, the two setups are *not* the same. The details of this fact are left for Problem 4, but the following observation should convince you that they are indeed different. Let the ball be released from rest in both scenarios at a large value of x. Then the force, kx, due to the spring is very large. But the force in the x-direction on the particle in the valley is only a fraction of mg (namely $mg\sin\theta\cos\theta$, where θ is the angle of the ground) at the start. ♣

4.2 Small Oscillations

Consider an object subject to a potential $V(x)$. Let the object initially be at rest at a local minimum of $V(x)$, and then let it be given a small kick so that it moves back and forth around the equilibrium point. What can we say about this motion? Is it harmonic? Does the frequency depend on the amplitude?

It turns out that for small amplitudes, the motion is indeed harmonic, and the frequency can easily be found, given $V(x)$. To see this, expand $V(x)$ in a Taylor series around the equilibrium point, x_0.

$$
V(x) = V(x_0)+V'(x_0)(x-x_0)+\frac{1}{2!}V''(x_0)(x-x_0)^2+\frac{1}{3!}V'''(x_0)(x-x_0)^3+\cdots. \tag{4.9}
$$

We can simplify this greatly. $V(x_0)$ is an irrelevant additive constant; since only differences in energy matter (or equivalently, since $F = -dV/dx$), we can ignore it. And $V'(x_0) = 0$, by definition of the equilibrium point. So that leaves us with the $V''(x_0)$ and higher order terms. For sufficiently small displacements, these higher order terms are negligible compared to the $V''(x_0)$ term, because they are suppressed by additional powers of $(x - x_0)$. (Even if $V'''(x_0)$ is much larger than $V''(x_0)$, we can always pick $(x - x_0)$ small enough so that the $V'''(x_0)$ term is negligible. The one case where this is not true is when $V''(x_0) = 0$. But the result in eq. (4.11) below is still correct in this case.) So we are left with

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2. \tag{4.10}$$

But this looks exactly like a Hooke's law potential, $V(x) = (1/2)k(x - x_0)^2$, if we let $V''(x_0)$ be our "spring constant", k. The frequency of (small) oscillations, $\omega = \sqrt{k/m}$, is therefore

$$\omega = \sqrt{\frac{V''(x_0)}{m}}\,. \tag{4.11}$$

This is an important result, because *any* function, $V(x)$, looks basically like a parabola (see Fig. 4.3) in a small enough region around a minimum (except for the special case when $V''(x_0) = 0$).

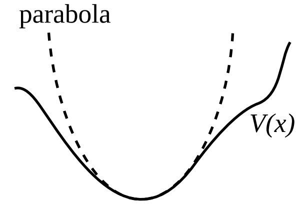


Figure 4.3

A potential may look quite erratic,
And its study may seem problematic.
But down near a min,
You can say with a grin,
"It behaves like a simple quadratic!"

4.3 Conservation of energy in 3-D

The concepts of work and potential energy in three dimensions are slightly more complicated than in one dimension, but the general ideas are the same. We start with $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ depends only on position, that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$. This vector equation is shorthand for three equations analogous to eq. (4.1), namely $mv_x(dv_x/dx) = F_x$, and likewise for y and z. Multiplying through by dx, etc., in these three equations, and then adding them together gives

$$F_x dx + F_y dy + F_z dz = m(v_x dv_x + v_y dv_y + v_z dv_z). \tag{4.12}$$

Integrating from the point (x_0, y_0, z_0) to the point (x, y, z) yields

$$E + \int_{x_0}^{x} F_x dx + \int_{y_0}^{y} F_y dy + \int_{z_0}^{z} F_z dz = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}mv^2, \tag{4.13}$$

where E is a constant of integration. (Technically, we should put primes on the integration variables, so that we don't confuse them with the limits of integration, but this gets too messy.) Note that the integrations on the left-hand side depend on what path in 3-D space is chosen to go from (x_0, y_0, z_0) to (x, y, z). We will address this issue below.

With $d\mathbf{r} \equiv (dx, dy, dz)$, the left-hand side of eq. (4.12) is equal to $\mathbf{F} \cdot d\mathbf{r}$. Hence, eq. (4.13) may be written as

$$\frac{1}{2}mv^2 - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = E. \tag{4.14}$$

Therefore, if we define the potential energy, $V(\mathbf{r})$, as

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.15}$$

then we may write

$$\frac{1}{2}mv^2 + V(\mathbf{r}) = E. \tag{4.16}$$

In other words, the sum of the kinetic energy and potential energy is constant.

4.3.1 Conservative forces in 3-D

There is one complication that arises in 3-D that we didn't have to worry about in 1-D. The potential energy defined in eq. (4.15) may not be well defined. That is, it may be path-dependent. In 1-D, there is only one way to get from x_0 to x. But in 3-D, there is an infinite number of paths that go from $\mathbf{r}_0$ to $\mathbf{r}$. In order for the potential, $V(\mathbf{r})$, to have any meaning and be of any use, it must be well-defined, that is, path-independent. A force for which this is the case is called a *conservative force*. Let us now see what types of forces are conservative.

Theorem 4.2 *Given a force,* $\mathbf{F}(\mathbf{r})$*, a necessary and sufficient condition for the potential,*

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.17}$$

to be well-defined (that is, path independent) is that the curl of $\mathbf{F}$ *is zero (that is,* $\nabla \times \mathbf{F} = \mathbf{0}$*).*

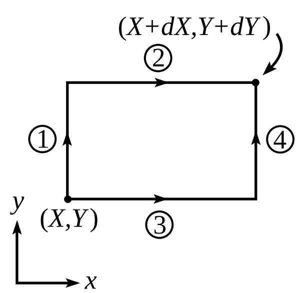


Figure 4.4

Proof: First, let us show that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path-independence.

Consider the infinitesimal rectangle shown in Fig. 4.4. (The rectangle lies in the x-y plane, so in the present analysis we will suppress the z-component of all coordinates, for convenience.) In order to have the potential be path-independent, the work done in going from (X,Y) to $(X+dX, Y+dY)$, namely the integral $\int \mathbf{F}\cdot d\mathbf{r}$, must be path-independent. In particular, the integral along the segments '1' and '2' must equal the integral along segments '3' and '4'. That is, $\int_1 F_y dy + \int_2 F_x dx = \int_3 F_x dx + \int_4 F_y dy$. Therefore, a necessary condition for path-independence is

$$\begin{aligned} \int_2 F_x dx - \int_3 F_x dx = \int_4 F_y dy - \int_1 F_y dy \qquad \Longrightarrow \\ \int_X^{X+dX} \Big(F_x(x, Y+dY) - F_x(x,Y)\Big) dx \\ = \int_Y^{Y+dY} \Big(F_y(X+dX, y) - F_y(X,y)\Big) dy. \end{aligned} \tag{4.18}$$

Now,

$$F_x(x, Y+dY) - F_x(x,Y) \approx dY \frac{\partial F_x(x,y)}{\partial y}\bigg|_{(x,Y)} \approx dY \frac{\partial F_x(x,y)}{\partial y}\bigg|_{(X,Y)}. \tag{4.19}$$

The first approximation holds due to the definition of the partial derivative. The second approximation holds because our rectangle is small enough so that x is essentially equal to X.

Similar treatment works for the F_y terms, so eq. (4.18) becomes

$$\int_X^{X+dX} dY \frac{\partial F_x(x,y)}{\partial y}\bigg|_{(X,Y)} dx = \int_Y^{Y+dY} dX \frac{\partial F_y(x,y)}{\partial x}\bigg|_{(X,Y)} dy. \tag{4.20}$$

The integrands here are constants, so we obtain

$$dX dY \left(\frac{\partial F_x(x,y)}{\partial y} - \frac{\partial F_y(x,y)}{\partial x}\right)\bigg|_{(X,Y)} = 0. \tag{4.21}$$

Canceling the $dXdY$ factor, and noting that (X,Y) is an arbitrary point, we see that if the potential is path-independent, then we must have

$$\frac{\partial F_x(x,y)}{\partial y} - \frac{\partial F_y(x,y)}{\partial x} = 0, \tag{4.22}$$

at any point (x,y).

The preceding analysis also works, of course, for little rectangles in the x-z and y-z planes, so we obtain two other similar conditions for the potential to be uniquely defined. All three conditions may be concisely written as

$$\nabla \times \mathbf{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0. \tag{4.23}$$

We have therefore shown that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path-independence. Let us now show that it is sufficient.

The proof of sufficiency follows immediately from Stokes' theorem, which states that (see Fig. 4.5)

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}. \tag{4.24}$$

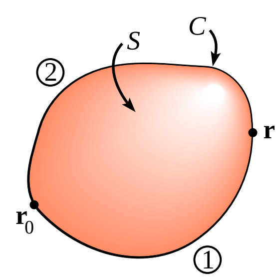


Figure 4.5

Here, C is an arbitrary closed curve, which we make pass through $\mathbf{r}_0$ and $\mathbf{r}$. S is an arbitrary surface which has C as its boundary. And $d\mathbf{A}$ has a magnitude equal to an infinitesimal piece of area on S and a direction defined to be orthogonal to S.

Therefore, if $\nabla \times \mathbf{F} = 0$ everywhere, then $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any closed curve. But Fig. 4.5 shows that traversing the loop C entails traversing path '1' in the 'forward' direction, and then traversing path '2' in the 'backward' direction. Hence, $\int_1 \mathbf{F} \cdot d\mathbf{r} - \int_2 \mathbf{F} \cdot d\mathbf{r} = 0$. Therefore, any two paths give the same integral, as we wanted to show. ■

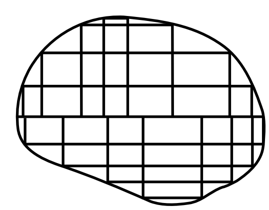


Figure 4.6

REMARK: If you don't like invoking Stokes' theorem, then you can just back up a step and prove it from scratch. Here's the rough idea of the proof. For simplicity, pick a path confined to the x-y plane (the general case proceeds in the same manner). For the purposes of integration, any path can be approximated by a series of little segments parallel to the coordinate axes (see Fig. 4.6).

Now imagine integrating $\int \mathbf{F} \cdot d\mathbf{r}$ over every little rectangle in the figure (in a counter-clockwise direction). The result may be viewed in two ways: (1) From the above analysis (leading to eq. (4.21)), each integral gives the curl times the area of the rectangle. So whole integral gives $\int_S (\nabla \times \mathbf{F}) dA$. (2) Each interior line gets counted twice (in opposite directions) in the whole integration, so these contributions cancel. We are left with the integral over the edge segments, which gives $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$. ♣

REMARK: Another way to show that $\nabla \times \mathbf{F} = 0$ is a necessary condition for path-independence (that is, "If $V(\mathbf{r})$ is well-defined, then $\nabla \times \mathbf{F} = 0$.") is the following.

If $V(\mathbf{r})$ is well-defined, then it is legal to write down the differential form of eq. (4.15). This is

$$dV(\mathbf{r}) = -\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \equiv -(F_x dx + F_y dy + F_z dz). \tag{4.25}$$

But another expression for dV is

$$dV(\mathbf{r}) = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{4.26}$$

The previous two equations must be equivalent for arbitrary dx, dy, and dz. So we have

$$\begin{aligned} (F_x, F_y, F_z) &= -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) \\ \Longrightarrow \qquad \mathbf{F}(\mathbf{r}) &= -\nabla V(\mathbf{r}). \end{aligned} \tag{4.27}$$

In other words, the force is simply the gradient of the potential. Therefore,

$$\nabla \times \mathbf{F} = -\nabla \times \nabla V(\mathbf{r}) = 0, \tag{4.28}$$

because the curl of a gradient is zero (as you can explicitly verify). ♣

Example (Central force): A *central force* is defined to be a force that points radially, and whose magnitude depends on only r. That is, $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$. A central force is a conservative force, as we will now verify by explicitly showing that $\nabla \times \mathbf{F} = \mathbf{0}$.

$\mathbf{F}$ may be written as

$$\mathbf{F}(x, y, z) = F(r)\hat{\mathbf{r}} = F(r) \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right). \tag{4.29}$$

Note that

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{x}{r}, \tag{4.30}$$

and similarly for y and z. The z component of $\nabla \times \mathbf{F}$ is therefore (writing F for $F(r)$, and F' for $dF(r)/dr$)

$$\begin{aligned} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial (yF/r)}{\partial x} - \frac{\partial (xF/r)}{\partial y} \\ &= \left(\frac{y}{r} F' \frac{\partial r}{\partial x} - yF \frac{1}{r^2} \frac{\partial r}{\partial x} \right) - \left(\frac{x}{r} F' \frac{\partial r}{\partial y} - xF \frac{1}{r^2} \frac{\partial r}{\partial y} \right) \\ &= \left(\frac{yxF'}{r^2} - \frac{yxF}{r^3} \right) - \left(\frac{xyF'}{r^2} - \frac{xyF}{r^3} \right) = 0. \end{aligned} \tag{4.31}$$

Likewise for the x- and y-components.

Alternatively, you could demonstrate that the potential is well-defined by simply verifying that the function

$$V(r) = -\int_{r_0}^{r} F(r')dr' \tag{4.32}$$

has its (negative) gradient equal to $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$.

An example of a nonconservative force is friction. A friction force is certainly not a function of only position; its sign changes depending on which way the object is moving.

4.4 Gravity due to a sphere

4.4.1 Derivation via the potential energy

We know that the gravitational force on a point-mass m, located a distance r from a point-mass M, is given by Newton's law of gravitation,

$$F(r) = \frac{-GMm}{r^2}, \tag{4.33}$$

where the minus sign indicates an attractive force. What is the force if we replace the latter point-mass by a sphere of radius R and mass M? The answer (as long as the sphere is spherically symmetric, that is, the density is a function of only r) is that it is still GMm/r^2. A sphere acts just like a point mass, for the purposes of gravity. This is an extremely pleasing result, to say the least. If it were not the case, then the universe would be a far more complicated place than it is. In particular, the motion of planets and such things would be rather hard to describe.

To prove the above claim, it turns out to be much easier to calculate the potential energy due to a sphere, and to then take the derivative to obtain the force, than to calculate the force explicitly. So this is the route we will take. It will suffice to demonstrate the result for a thin spherical shell, because a sphere is the sum of many such shells.

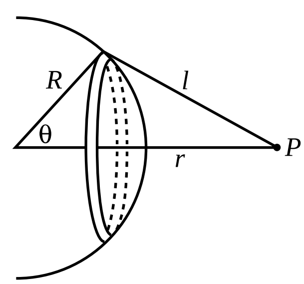


Figure 4.7

Our strategy in calculating the potential energy, at a point P, due to a spherical shell will be to slice the shell into rings as show in Fig. 4.7. Let the radius of the shell be R, and let P be a distance r from the center of the shell. Let the ring make the angle θ as shown, and let P be a distance ℓ from the ring.

The length ℓ is a function of R, r, and θ. It may be found as follows. In Fig. 4.8, segment AB has length $R\sin\theta$, and segment BP has length $r - R\cos\theta$. So the length ℓ in triangle ABP is

$$\ell = \sqrt{(R\sin\theta)^2 + (r - R\cos\theta)^2} = \sqrt{R^2 + r^2 - 2rR\cos\theta}. \tag{4.34}$$

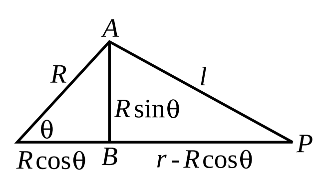


Figure 4.8

This, of course, is just the law of cosines.

The area of a ring between θ and $\theta + d\theta$ is its width (which is $Rd\theta$) times its circumference (which is $2\pi R\sin\theta$). Letting $\sigma = M/(4\pi R^2)$ be the mass density of the shell, the potential energy of a mass m at P due to a thin ring is therefore $-Gm\sigma(Rd\theta)(2\pi R\sin\theta)/\ell$. This is true because the gravitational potential energy,

$$V(r) = \frac{-Gm_1m_2}{r}, \tag{4.35}$$

is a scalar quantity, so the contributions from little mass pieces simply add. The total potential energy at P is therefore

$$\begin{aligned} V(r) &= -\int_0^\pi \frac{2\pi\sigma GR^2 m\sin\theta d\theta}{\sqrt{R^2+r^2-2rR\cos\theta}} \\ &= -\frac{2\pi\sigma GRm}{r}\sqrt{R^2+r^2-2rR\cos\theta}\,\Big|_0^\pi . \end{aligned} \tag{4.36}$$

(As messy as the integral looks, the $\sin\theta$ in the numerator is what makes it easily doable.)

There are two cases to consider. If $r > R$, then we have

$$V(r) = -\frac{2\pi\sigma GRm}{r}\Big((r+R)-(r-R)\Big) = -\frac{G(4\pi\sigma R^2)m}{r} = -\frac{GMm}{r}, \tag{4.37}$$

which is the potential due to a point-mass M, as promised. If $r < R$, then we have

$$V(r) = -\frac{2\pi\sigma GRm}{r}\Big((r+R)-(R-r)\Big) = -\frac{G(4\pi\sigma R^2)m}{R} = -\frac{GMm}{R}, \tag{4.38}$$

which is independent of r.

Having found $V(r)$, we now simply have to take the negative of its gradient to obtain $F(r)$. The gradient is just $\hat{\mathbf{r}}(d/dr)$ here, because V is a function of only r, so we have

$$\begin{aligned} F(r) &= -\frac{GMm}{r^2}, \qquad \text{if } r > R, \\ F(r) &= 0, \qquad \text{if } r < R. \end{aligned} \tag{4.39}$$

These forces are directed radially, of course. A sphere is the sum of many spherical shells, so if P is outside a given sphere, then the force at P is GMm/r^2, where M is the total mass of the sphere. The shells may have different mass densities (but each one must have uniform density), and this result will still hold.

> Newton looked at the data, numerical,
> And then observations, empirical.
> He said, "But, of course,
> We get the same force
> From a point mass and something that's spherical!"

If P is inside a given sphere, then the only relevant material is the mass inside a concentric sphere through P, because all the shells outside this region give zero force, from the second equation in eq. (4.39). The material 'outside' of P is, for the purposes of gravity, not there.

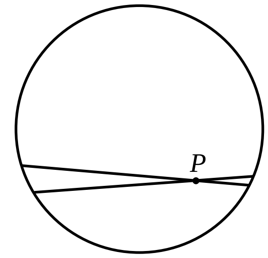


Figure 4.9

It is not obvious that the force inside a spherical shell is zero. Consider the point P in Fig. 4.9. A piece of mass, dm, on the right side of the shell gives a larger force on P than a piece of mass, dm, on the left side, due to the $1/r^2$ dependence. But there is more mass on the left side than the right side. These two effects happen to exactly cancel, as you can show in Problem 9.

Note that the gravitational force between two spheres is the same as if they are replaced by two point-masses. This follows from two applications of the result of this section.

4.4.2 Tides

The tides on the earth are due to the fact that the gravitational field from a point mass (or a spherical object) is not uniform. The direction of the force is not constant (the field lines converge to the source), and the magnitude is not constant (it falls off like $1/r^2$). As far as the earth goes, these effects (mainly due to the gravitational force from the moon, and not the sun, as we will show below) cause the oceans to bulge around the earth, producing the observed tides.

The study of tides here is useful partly because tides are a very real phenomenon in this world, and partly because the following analysis gives us an excuse to make lots of neat mathematical approximations. Before considering the general case of tidal forces, let's look at two special cases.

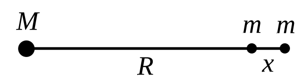


Figure 4.10

- **Longitudinal tidal force**

  In Fig. 4.10, two particles of mass m are located at points $(R, 0)$ and $(R+x, 0)$, with $x \ll R$. A planet of mass M is located at the origin. What is the difference between the gravitational forces acting on these two masses?

  The difference in the forces is (using $x \ll R$ to make suitable approximations)

$$\frac{-GMm}{R^2} - \frac{-GMm}{(R+x)^2} \approx \frac{-GMm}{R^2} + \frac{GMm}{R^2 + 2Rx} = \frac{GMm}{R^2}\left(-1 + \frac{1}{1 + 2x/R}\right)$$
$$\approx \frac{GMm}{R^2}\Big(-1 + (1 - 2x/R)\Big) = \frac{-2GMmx}{R^3}. \quad (4.40)$$

  This is, of course, simply the derivative of the force, times x. This difference points along the line joining the masses, and its effect is to push the masses apart.

  We see that this force difference is linear in the separation, x, and inversely proportional to the cube of the distance from the source. This *force difference* is the important quantity (as opposed to the *force on each mass*) when we are dealing with the relative motion of objects in free-fall around a given mass (e.g., circular orbiting motion, or radial falling motion). If we take persons A and B to be our two masses, then neither can feel the gravitational force acting on them (for all they know, they are floating freely in space). But if they are connected with a string, then each will feel a tension in the string equal to $2GMmx/R^3$ (neglecting higher-order terms in x/R). If they are placed in a windowless box, this is the only force they will be able to observe.

Instead of the phrase "force difference", the term *tidal force* is generally used. From the point of view of the falling people, this is actually the better way of looking at the situation. They don't observe the net gravitational field, so it doesn't make much sense for them to take the difference between the values of the force.

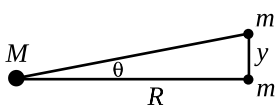


Figure 4.11

- **Transverse tidal force**

  In Fig. 4.11, two particles of mass m are located at points $(R, 0)$ and (R, y), with $y \ll R$. A planet of mass M is located at the origin. What is the difference between the gravitational forces acting on these two masses?

  Both masses are the same distance R from the origin (up to second-order effects in y/R), so the magnitudes of the forces on them are essentially the same. The direction is the only thing that is different, to first order in y/R. The difference in the forces is the y-component of the force on the top mass. The magnitude of this component is

$$\frac{GMm}{R^2}\sin\theta \approx \frac{GMm}{R^2}\left(\frac{y}{R}\right) = \frac{GMmy}{R^3}. \tag{4.41}$$

  This difference points along the line joining the masses, and its effect is to pull the masses together. As in the longitudinal case, the transverse tidal force is linear in the separation, y, and inversely proportional to the cube of the distance from the source.

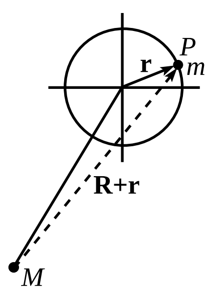


Figure 4.12

Let us now calculate the tidal force at an arbitrary point on a circle of radius r, centered at the origin. We'll calculate the tidal force relative to the origin. Let the source of the gravitational force be a mass M located at the vector $-\mathbf{R}$ (so that the vector from the source to a point P on the circle is $\mathbf{R} + \mathbf{r}$; see Fig. 4.12). As usual, assume $r \ll |\mathbf{R}|$.

The attractive gravitational force may be written as $\mathbf{F}(\mathbf{x}) = -GMm\mathbf{x}/|\mathbf{x}|^3$, where $\mathbf{x}$ is the vector from the source to the point in question. (The cube is in the denominator because the vector in the numerator contains one power of the distance.) The desired difference between the force on a mass m at P and the force on a mass m at the origin is the tidal force,

$$\frac{\mathbf{F}_{\mathrm{t}}(\mathbf{r})}{GMm} = \frac{-(\mathbf{R}+\mathbf{r})}{|\mathbf{R}+\mathbf{r}|^3} - \frac{-\mathbf{R}}{|\mathbf{R}|^3}. \tag{4.42}$$

This is the exact expression for the tidal force; it doesn't get any more correct than this. However, it is completely useless. Although a mathematician might be quite at home in this state of correctness, we physicists must begin fiddling with the result. Let's make some approximations in eq. (4.42) and transform it into something incorrect (as most approximations tend to be), but far more useful.

The first thing we have to do is rewrite the $|\mathbf{R} + \mathbf{r}|$ term. We have (using $r \ll R$ to ignore higher-order terms)

$$|\mathbf{R}+\mathbf{r}| \;=\; \sqrt{(\mathbf{R}+\mathbf{r})\cdot(\mathbf{R}+\mathbf{r})}$$

$$
\begin{aligned}
&= \sqrt{R^2 + r^2 + 2\mathbf{R}\cdot\mathbf{r}} \\
&\approx R\sqrt{1 + 2\mathbf{R}\cdot\mathbf{r}/R^2} \\
&\approx R\left(1 + \frac{\mathbf{R}\cdot\mathbf{r}}{R^2}\right).
\end{aligned} \tag{4.43}
$$

Therefore (using $r \ll R$),

$$
\begin{aligned}
\frac{\mathbf{F}_{\rm t}(\mathbf{r})}{GMm} &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+\mathbf{R}\cdot\mathbf{r}/R^2)^3} + \frac{\mathbf{R}}{R^3} \\
&\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+3\mathbf{R}\cdot\mathbf{r}/R^2)} + \frac{\mathbf{R}}{R^3} \\
&\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3}\left(1 - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\right) + \frac{\mathbf{R}}{R^3}.
\end{aligned} \tag{4.44}
$$

Letting $\hat{\mathbf{R}} \equiv \mathbf{R}/R$, we finally have (using $r \ll R$)

$$
\mathbf{F}_{\rm t}(\mathbf{r}) \approx \frac{GMm(3\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\mathbf{r}) - \mathbf{r})}{R^3}. \tag{4.45}
$$

If we let M lie on the negative x-axis, so that $\hat{\mathbf{R}} = \hat{\mathbf{x}}$, then $\hat{\mathbf{R}}\cdot\mathbf{r} = x$, and we see that the tidal force at the point $P = (x, y)$ may be written as

$$
\mathbf{F}_{\rm t}(\mathbf{r}) \approx \frac{GMm(2x, -y)}{R^3}. \tag{4.46}
$$

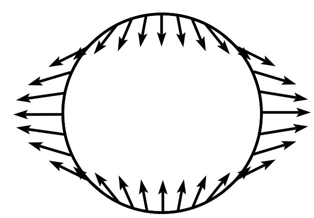


Figure 4.13

This clearly reduces properly in the two special cases considered above. The tidal forces at various points on the circle are shown in Fig. 4.13.

If the earth were a rigid body, then these forces would be irrelevant. But the water in the oceans is free to slosh around. The water on the earth bulges toward the moon. As the earth rotates, the bulge moves around relative to the earth. This produces two high tides and two low tides per day. (It's not exactly two per day, because the moon moves around the earth. But this motion is fairly slow, taking about a month, so it's a decent approximation for the present purposes to think of the moon as motionless.)

Note that it is *not* the case, of course, that the moon *pushes* the water away on the far side of the earth. It pulls on that water, too; it just does so in a weaker manner than it pulls on the rigid part of the earth. Tides are a *comparative* effect.

REMARK: Consider two 1 kg masses on the earth separated by a distance of 1 m. An interesting fact is that the gravitational force from the sun on them is (much) larger than that from the moon. But the tidal force from the sun on them is (slightly) weaker than that from the moon. The forces are

$$
\begin{aligned}
F_S &= \frac{GM_S}{R^2_{E,S}} \approx 6\cdot 10^{-3}\,\mathrm{kg\,m/s^2}, \\
F_M &= \frac{GM_M}{R^2_{E,M}} \approx 3.4\cdot 10^{-5}\,\mathrm{kg\,m/s^2}.
\end{aligned} \tag{4.47}
$$

And the tidal forces are roughly

$$F_{\text{t,S}} \sim \frac{GM_S}{R_{E,S}^3} \approx 4 \cdot 10^{-14} \text{kg m/s}^2,$$
$$F_{\text{t,M}} \sim \frac{GM_M}{R_{E,M}^3} \approx 9 \cdot 10^{-14} \text{kg m/s}^2. \quad (4.48)$$

The moon's effect is roughly twice the sun's. ♣

4.5 Conservation of linear momentum

4.5.1 Conservation of p

Newton's third law says that for every force there is an equal and opposite force. More precisely, if $\mathbf{F}_{ab}$ is the force that particle a feels from particle b, and $\mathbf{F}_{ba}$ is the force that particle b feels from particle a, then $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$, at any time.[2]

This has important implications concerning momentum. Consider two particles that interact over a period of time. Assume they are isolated from outside forces. Because

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad (4.49)$$

the total change in a particle's momentum is the time integral of the force on that particle (this integral is called the *impulse*). That is,

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} \mathbf{F} dt. \quad (4.50)$$

Therefore, since $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$ at all times, we have

$$\begin{aligned} \mathbf{p}_a(t_2) - \mathbf{p}_a(t_1) &= \int_{t_1}^{t_2} \mathbf{F}_{ab} dt \\ &= -\int_{t_1}^{t_2} \mathbf{F}_{ba} dt = -\Big(\mathbf{p}_b(t_2) - \mathbf{p}_b(t_1)\Big), \end{aligned} \quad (4.51)$$

and so

$$\mathbf{p}_a(t_1) + \mathbf{p}_b(t_1) = \mathbf{p}_a(t_2) + \mathbf{p}_b(t_2). \quad (4.52)$$

In other words, the total momentum of this isolated system is *conserved*. It does not depend on time.

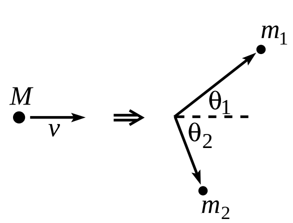


Figure 4.14

Example (Splitting mass): A mass M moves with speed V in the x-direction. It explodes into two pieces that go off at angles θ_1 and θ_2, as shown in Fig. 4.14. What are the magnitudes of the momenta of the two pieces?

[2]Some forces, such as magnetic forces from moving charges, do not satisfy the third law. But for any common 'pushing' or 'pulling' force (the type we will deal with), the third law holds.

Solution: Let $P \equiv MV$ be the initial momentum, and let p_1 and p_2 be the final momenta. Conservation of momentum in the x- and y-directions gives, respectively,

$$\begin{aligned} p_1 \cos\theta_1 + p_2 \cos\theta_2 &= P, \\ p_1 \sin\theta_1 - p_2 \sin\theta_2 &= 0. \end{aligned} \tag{4.53}$$

Solving for p_1 and p_2 (and using a trig addition formula) gives

$$p_1 = \frac{P \sin\theta_2}{\sin(\theta_1 + \theta_2)}, \qquad \text{and} \qquad p_2 = \frac{P \sin\theta_1}{\sin(\theta_1 + \theta_2)}. \tag{4.54}$$

A few limits: If $\theta_1 = \theta_2$, then $p_1 = p_2$, as it should. If, in addition, θ_1 and θ_2 are both small, then $p_1 = p_2 \approx P/2$, as they should. If, on the other hand, $\theta_1 = \theta_2 \approx 90°$, then p_1 and p_2 are very large; the explosion must have provided a large amount of energy.

REMARK: Newton's third law makes a statement about forces. But force is defined in terms of momentum via $F = dp/dt$. So the third law essentially *postulates* conservation of momentum. (The "proof" above in eq. (4.51) is hardly a proof; it involves one simple integration). So you might wonder if momentum conservation is something you can *prove*, or if it's something you have to *assume* (as we have basically done).

The difference between a postulate and a theorem is rather nebulous. One person's postulate might be another person's theorem, and vice-versa. You have to start *somewhere* in your assumptions. We choose to start with the third law. In the Lagrangian formalism in Chapter 5, the starting point is different, and momentum conservation is deduced as a consequence of translational invariance (as we will see). So it looks more like a theorem in that formalism.

But one thing is certain. Momentum conservation of two particles can *not* be proven from scratch for arbitrary forces, because it is *not true*. For example, if two charged particles interact in a certain way through the magnetic fields they produce, then the total momentum of the two particles may *not* be conserved. Where is the missing momentum? It is carried off in the electromagnetic field. The total momentum of the system *is* conserved, but the point is that the system consists of the two particles *plus* the electromagnetic field.

For normal, everyday pushing and pulling forces, simple arguments can be made to justify conservation of momentum. But some forces (e.g., the magnetic force) act in a 'sideways' manner. Newton's third law does not necessarily hold for particles subject to such forces. ♣

Now let's look at momentum conservation for a system of many particles. As above, let $\mathbf{F}_{ij}$ be the force that particle i feels from particle j. Then $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, at any time. Assume the particles are isolated from outside forces.

The change in the momentum of the ith particle, from t_1 to t_2 (we won't bother writing all the t's in the expressions below), is

$$\Delta \mathbf{p}_i = \int \left(\sum_j \mathbf{F}_{ij} \right) dt. \tag{4.55}$$

Therefore, the change in the total momentum of all the particles is

$$\Delta \mathbf{P} \equiv \sum_i \Delta \mathbf{p}_i = \int \left(\sum_i \sum_j \mathbf{F}_{ij} \right) dt. \tag{4.56}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ at all times, because for every term $\mathbf{F}_{ab}$ there is a term $\mathbf{F}_{ba}$, and $\mathbf{F}_{ab} + \mathbf{F}_{ba} = 0$. (And also, $\mathbf{F}_{aa} = 0$.) Therefore, the total momentum of an isolated system of particles is conserved.

4.5.2 Rocket motion

The application of momentum conservation becomes a little more exciting when the mass, m, is allowed to vary. Such is the case with rockets, since most of their mass consists of fuel which is eventually ejected.

Let mass be ejected with speed u relative to the rocket,[3] at a rate dm/dt. We'll define the quantity dm to be negative; so during a time dt the mass dm gets *added* to the rocket's mass. (If you wanted, you could define dm to be positive, and then *subtract* it from the rocket's mass. Either way is fine.) Also, we'll define u to be positive; so the ejected particles *lose* a speed u relative to the rocket. It may sound silly, but the hardest thing about rocket motion is picking a sign for these quantities and sticking with it.

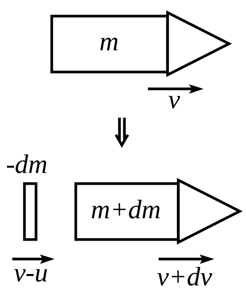


Figure 4.15

Consider a moment when the rocket has mass m and speed v. Then at a time dt later (see Fig. 4.15), the rocket has mass $m+dm$ and speed $v+dv$, while the exhaust has mass $(-dm)$ and speed $v-u$ (which may be positive or negative, depending on the relative size of v and u). There are no external forces, so the total momentum at each of these times must be equal. Therefore,

$$mv = (m + dm)(v + dv) + (-dm)(v - u). \tag{4.57}$$

Ignoring the second-order term yields $m\,dv = -u\,dm$. Dividing by m and integrating from t_1 to t_2 gives

$$dv = -u\frac{dm}{m} \qquad \Longrightarrow \qquad v_2 - v_1 = u \ln \frac{m_1}{m_2}. \tag{4.58}$$

For the case where the initial mass is M and the initial speed is 0, we have $v = u\ln(M/m)$. And if dm/dt is equal to the constant (negative) number η, then $v(t) = u\ln[M/(M+\eta t)]$.

The log in the result in eq. (4.58) is not very encouraging. If the mass of the metal of the rocket is m, and the mass of the fuel is $9m$, then the final speed is only $u \ln 10$. If the mass of the fuel is increased by a factor of 11 up to $99m$ (which is probably not even structurally possible, given the amount of metal required to hold it), then the final speed only doubles to $u\ln 100 = 2(u \ln 10)$. How do you make a rocket go significantly faster? Exercise 6 deals with this question.

[3] Just to emphasize, u is the speed with respect to the rocket. It wouldn't make much sense to say "relative to the ground", because the rocket's engine spits out the matter relative to itself, and the engine has no way of knowing how fast the rocket is moving with respect to the ground.

4.6 The CM frame

4.6.1 Definition

When talking about momentum, it is tacitly assumed that a certain frame of reference has been picked; the velocities of the particles have to be measured with respect to some coordinate system. Any inertial (that is, non-accelerating) frame is as good as any other, but we will see that there is one particular reference frame that is usually advantageous to use.

Consider a frame S, and another frame S' which moves at constant velocity $\mathbf{u}$ with respect to S (see Fig. 4.16). Given a system of particles, the velocity of the ith particle in S' is related to its velocity in S by

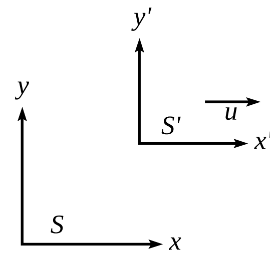


Figure 4.16

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}. \tag{4.59}$$

It is then easy to see that if momentum is conserved during a collision in frame S, then it is also conserved in frame S'. This is true because both the initial and final momenta of the system in S' are decreased by the same amount, $(\sum m_i)\mathbf{u}$, compared to what they were in S.[4]

Let us therefore consider the unique frame in which the total momentum is zero. This is called the *center-of-mass frame*, or CM frame. If the total momentum is $\mathbf{P} \equiv \sum m_i \mathbf{v}_i$ in frame S, then it is easy to see that the CM frame, S', is the frame that moves with speed

$$\mathbf{u} = \frac{\mathbf{P}}{\sum m_j} \equiv \frac{\sum m_i \mathbf{v}_i}{\sum m_j} \tag{4.60}$$

with respect to S. This is true because

$$\begin{aligned} \mathbf{P}' &= \sum m_i \mathbf{v}'_i \\ &= \sum m_i \left(\mathbf{v}_i - \frac{\mathbf{P}}{\sum m_j} \right) \\ &= \mathbf{P} - \mathbf{P} = \mathbf{0}, \end{aligned} \tag{4.61}$$

as desired. The CM frame is extremely useful. Physical processes are generally much more symmetrical in this frame, and this makes the results more transparent. Also, the kinetic energy takes a very nice form in the CM frame, as we will see shortly.

REMARK: The CM frame could also be called the "zero-momentum" frame. The "CM" name is used because the center-of-mass of the particles (defined by $\mathbf{R}_{\mathrm{CM}} \equiv \sum m_i \mathbf{r}_i / \sum m_j$, which is the location of the pivot upon which the particles would balance, if they were rigidly connected) does not move. This is obvious from

$$\frac{d\mathbf{R}_{\mathrm{CM}}}{dt} = \frac{1}{\sum m_j} \sum m_i \frac{d\mathbf{r}_i}{dt} \propto \sum \mathbf{p}_i = 0. \tag{4.62}$$

[4] Alternatively, nowhere in our earlier derivation of momentum conservation did we say what frame we were using. We only assumed that the frame is not accelerating. If it *is* accelerating, then $\mathbf{F}$ does *not* equal $m\mathbf{a}$ (where $\mathbf{F}$ is the force a particle exerts on another). We will see in Chapter 9 how $\mathbf{F} = m\mathbf{a}$ is modified in a non-inertial frame.

The center-of-mass may therefore be chosen as the origin of the CM frame. ♣

The other frame which people generally work with is the *lab frame*. There is nothing at all special about this frame. It is simply the frame (assumed to be inertial) in which the conditions of the problem are given. Any inertial frame can be called the "lab frame". Part of the task of many problems is to switch back and forth between the lab and CM frames. For example, if the final answer is requested in the lab frame, then you may want to transform the given information from the lab frame into the CM frame where things are more obvious, and then transform back to the lab frame to give the answer.

Figure 4.17

Example (Two masses in 1-D): A mass m with speed v approaches a stationary mass M (see Fig. 4.17). The masses bounce off each other without losing any energy. What are the final speeds of the particles? (Assume all motion takes place in 1-D.)

Solution: Doing this problem in the lab frame would require a potentially messy use of conservation of energy. But if we work in the CM frame, things are much easier. The total momentum in the lab frame is mv, so the CM frame moves to the right at speed $mv/(m+M)$ with respect to the lab frame. Therefore, in the CM frame, the speeds of the two masses are

$$v_m = \frac{Mv}{m+M}, \qquad \text{and} \qquad v_M = -\frac{mv}{m+M}. \tag{4.63}$$

These speeds are of course in the ratio M/m, and their difference is v.

In the CM frame, the two particles must simply reverse their velocities after the collision (provided they do indeed hit each other). This is true because the speeds must still be in the ratio M/m after the collision (so that the total momentum is still zero). Therefore, they must either both increase or both decrease. But if they do either of these, then energy is not conserved.[5]

If we now go back to the lab frame by adding the CM speed of $mv/(m+M)$ to the two new speeds of $-Mv/(m+M)$ and $mv/(m+M)$, we obtain final lab speeds of

$$v_m = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}. \tag{4.64}$$

NOTE: If $m = M$, then the left mass stops, and the right mass picks up a speed of v. If $M \gg m$, then the left mass bounces back with speed $\approx v$, and the right mass hardly moves. If $m \gg M$, then the left mass keeps plowing along at speed $\approx v$, and the right mass picks up a speed of $\approx 2v$. This $2v$ is an interesting result (it is clear if you consider the frame of the heavy mass m) which leads to some neat effects, as in Problem 24.

4.6.2 Kinetic energy

Given a system of particles, the relationship between the total kinetic energy in two different frames is generally rather messy and unenlightening. But if one of the frames is the CM frame, then the relationship turns out to be quite nice.

[5] So we *did* have to use conservation of energy, but in a far less messy way than the lab frame would have entailed.

Let S' be the CM frame, which moves at constant velocity $\mathbf{u}$ with respect to another frame S. Then the velocities in the two frames are related by

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{4.65}$$

The kinetic energy in the CM frame is

$$\mathrm{KE}_{\mathrm{CM}} = \frac{1}{2}\sum m_i |\mathbf{v}'_i|^2. \tag{4.66}$$

The kinetic energy in frame S is

$$\begin{aligned} \mathrm{KE}_{\mathrm{S}} &= \frac{1}{2}\sum m_i |\mathbf{v}'_i + \mathbf{u}|^2 \\ &= \frac{1}{2}\sum m_i (\mathbf{v}'_i \cdot \mathbf{v}'_i + 2\mathbf{v}'_i \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}) \\ &= \frac{1}{2}\sum m_i |\mathbf{v}'_i|^2 + \mathbf{u} \cdot \left(\sum m_i \mathbf{v}'_i\right) + \frac{1}{2}|\mathbf{u}|^2 \sum m_i \\ &= \mathrm{KE}_{\mathrm{CM}} + \frac{1}{2}M|\mathbf{u}|^2, \end{aligned} \tag{4.67}$$

where M is the total mass of the system, and where we have used $\sum_i m_i \mathbf{v}'_i = 0$, by definition of the CM frame. Therefore, the KE in any frame equals the KE in the CM frame, plus the kinetic energy of the whole system treated like a point mass M located at the CM. An immediate corollary of this is that if the KE is conserved in a collision in one frame, then it is conserved in any other frame.

4.7 Collisions

There are two basic types of collisions among particles, namely *elastic* ones (in which kinetic energy is conserved), and *inelastic* ones (in which kinetic energy is lost). In any collision, the total energy is conserved, but in inelastic collisions some of this energy goes into the form of heat (that is, relative motion of the atoms inside the particle) instead of showing up in the net translational motion of the particle.

We'll deal mainly with elastic collisions here (although some situations are inherently inelastic, as we'll discuss in Section 4.8). For inelastic collisions where it is stated that a certain fraction of the kinetic energy is lost, say 20%, only a trivial modification of the following procedures is required.

To solve any elastic collision problem, you simply have to write down the conservation of energy and momentum equations, and then solve for whatever variables you want to find.

4.7.1 1-D motion

Let's first look at one-dimensional motion. To see the general procedure, we'll solve the example in the previous section again.

Figure 4.18

Example (Two masses in 1-D, again): A mass m with speed v approaches a stationary mass M (see Fig. 4.18). The masses bounce off each other elastically. What are the final speeds of the particles? (Assume all motion takes place in 1-D.)

Solution: Let v' and V' be the final speeds of the masses. Then conservation of momentum and energy give, respectively,

$$\begin{aligned} mv + 0 &= mv' + MV', \\ \frac{1}{2}mv^2 + 0 &= \frac{1}{2}mv'^2 + \frac{1}{2}MV'^2. \end{aligned} \tag{4.68}$$

We must solve these two equations for the two unknowns v' and V'. Solving for V' in the first equation and substituting into the second gives

$$\begin{aligned} mv^2 &= mv'^2 + M\frac{m^2(v-v')^2}{M^2}, \\ \Longrightarrow \quad 0 &= (m+M)v'^2 - 2mvv' + (m-M)v^2, \\ \Longrightarrow \quad 0 &= \Big((m+M)v' - (m-M)v\Big)(v'-v). \end{aligned} \tag{4.69}$$

The $v' = v$ root is obvious, yet useless. $v' = v$ is of course a solution, because the initial conditions certainly satisfy conservation of energy and momentum with the initial conditions (how's that for a tautology). If you want, you can view $v' = v$ as the solution for the case where the particles miss each other. The fact that $v' = v$ is always a root can often save you a lot of quadratic-formula trouble.

The other root is the one we want. Putting this v' back into the first of eqs. (4.68) to obtain V' gives

$$v' = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad V' = \frac{2mv}{m+M}. \tag{4.70}$$

This solution was somewhat of a pain, because it involved quadratic equations. The following theorem is extremely useful because it offers a way to avoid the hassle of quadratic equations when dealing with 1-D elastic collisions.

Theorem 4.3 *In a 1-D elastic collision, the relative velocity of two particles after a collision is the negative of the relative velocity before the collision.*

Proof: Let the masses be m and M. Let v_i and V_i be the initial speeds. Let v_f and V_f be the final speeds. Conservation of momentum and energy give

$$\begin{aligned} mv_i + MV_i &= mv_f + MV_f \\ \frac{1}{2}mv_i^2 + \frac{1}{2}MV_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2. \end{aligned} \tag{4.71}$$

Rearranging these yields

$$\begin{aligned} m(v_i - v_f) &= M(V_f - V_i). \\ m(v_i^2 - v_f^2) &= M(V_f^2 - V_i^2) \end{aligned} \tag{4.72}$$

Dividing the second equation by the first gives $v_i + v_f = V_i + V_f$. Therefore,

$$v_i - V_i = -(v_f - V_f), \tag{4.73}$$

as was to be shown.

Note that in taking the quotient of these two equations, we have lost the $v_f = v_i$ and $V_f = V_i$ solution. But, as stated in the above example, this is the trivial solution. ■

This is a splendid theorem. It has the quadratic energy-conservation statement built into it. Hence, using the theorem along with momentum conservation (both of which are linear statements) gives the same information as the standard combination of eqs. (4.71).

Note that the theorem is quite obvious in the CM frame (as we argued in the example in Section 4.6.1). Therefore, it is true in any frame, since it involves differences between velocities.

4.7.2 2-D motion

Let's now look at the more general case of two-dimensional motion. (3-D motion is just more of the same, so we'll confine ourselves to 2-D.) Everything is basically the same as is 1-D, except that there is one more momentum equation, and one more variable to solve for. This is best seen through an example.

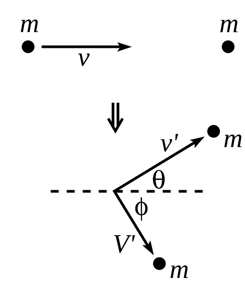


Figure 4.19

Example (Two masses in 2-D): A billiard ball with speed v approaches an identical stationary one. The balls bounce off each other elastically, in such a way that the incoming one gets deflected by an angle θ (see Fig. 4.19). What are the final speeds of the balls? What is the angle, ϕ, at which the stationary one is ejected?

Solution: Let v' and V' be the final speeds of the balls. Then conservation of p_x, p_y, and E give, respectively,

$$\begin{aligned} mv &= mv'\cos\theta + mV'\cos\phi, \\ mv'\sin\theta &= mV'\sin\phi, \\ \frac{1}{2}mv^2 &= \frac{1}{2}mv'^2 + \frac{1}{2}mV'^2. \end{aligned} \tag{4.74}$$

We must solve these three equations for the three unknowns v', V', and ϕ. There are various ways to do this. Here is one. Eliminate ϕ by adding the squares of the first two equations (after putting the $mv'\cos\theta$ on the left-hand side) to obtain

$$v^2 - 2vv'\cos\theta + v'^2 = V'^2. \tag{4.75}$$

Now eliminate V' by combining this with the third equation to obtain[6]

$$v' = v\cos\theta. \tag{4.76}$$

[6]Another solution is $v' = 0$. In this case, ϕ must equal zero, and θ is not well-defined. We simply have the 1-D motion of the example in section 4.6.1.

The third equation then implies

$$V' = v\sin\theta. \tag{4.77}$$

The second equation then gives $m(v\cos\theta)\sin\theta = m(v\sin\theta)\sin\phi$, which implies $\cos\theta = \sin\phi$, or

$$\phi = 90^\circ - \theta. \tag{4.78}$$

In other words, the balls bounce off at right angles with respect to each other. This fact is well known to pool players. Problem 19 gives another (cleaner) way to demonstrate this result.

4.8 Inherently inelastic processes

There is a nice class of problems where the system has inherently inelastic properties, even if it doesn't appear so at first glance. In such a problem, no matter how you try to set it up, there will be an inevitable mechanical energy loss that shows up in the form of heat. Total energy is conserved, of course; heat is simply another form of energy. But the point is that if you try to write down a bunch of $(1/2)mv^2$'s and conserve their sum, then you're going to get the wrong answer. The following example is the classic illustration of this type of problem.

Example (Sand on conveyor belt): Sand drops vertically at a rate σ kg/s onto a moving conveyor belt.

(a) What force must you apply to the belt in order to keep it moving at a constant speed v?
(b) How much kinetic energy does the sand gain per unit time?
(c) How much work do you do per unit time?
(d) How much energy is lost to heat per unit time?

Solution:

(a) Your force equals the rate of change of momentum. If we let m be the combined mass of conveyor belt plus sand on the belt, then

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = 0 + \sigma v, \tag{4.79}$$

where we have used the fact that v is constant.

(b) The kinetic energy gained per unit time is

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = \frac{dm}{dt}\left(\frac{v^2}{2}\right) = \frac{\sigma v^2}{2}. \tag{4.80}$$

(c) The work done by the force per unit time is

$$\frac{d(\text{Work})}{dt} = \frac{F\,dx}{dt} = Fv = \sigma v^2, \tag{4.81}$$

where we have used eq. (4.79).

(d) If work is done at a rate σv^2, and kinetic energy is gained at a rate $\sigma v^2/2$, then the "missing" energy must be lost to heat at a rate $\sigma v^2 - \sigma v^2/2 = \sigma v^2/2$.

In this example, it turned out that exactly the same amount of energy was lost to heat as was converted into kinetic energy of the sand. There is an interesting and simple way to see why this is true. In the following explanation, we'll just deal with one particle of mass M that falls onto the conveyor belt, for simplicity.

In the lab frame, the mass simply gains a kinetic energy of $Mv^2/2$ (by the time it finally comes to rest with respect to the belt), because the belt moves at speed v.

Now look at things in the conveyor belt's frame of reference. In this frame, the mass comes flying in with an initial kinetic energy of $Mv^2/2$, and then it eventually slows down and comes to rest on the belt. Therefore, all of the $Mv^2/2$ energy is converted to heat. And since the heat is the same in both frames, this is the amount of heat in the lab frame, too.

We therefore see that in the lab frame, the equality of the heat loss and the gain in kinetic energy is a consequence of the obvious fact that the belt moves at the same rate with respect to the lab (namely, v) as the lab moves with respect to the belt (also v).

In the solution in the above example, we did not assume anything about the nature of the friction force between the belt and the sand. The loss of energy to heat is an unavoidable result. You might think that if the sand comes to rest on the belt very "gently" (over a long period of time), then you can avoid the heat loss. This is not the case. In this scenario, the smallness of the friction force is compensated by the fact that the force must act over a very large distance. Likewise, if the sand comes to rest on the belt very abruptly, then the largeness of the force compensates for the smallness of the distance over which it acts. No matter how you set things up, the work done by the force (that is, the force times the distance over which it acts) is the same nonzero quantity.

Other problems of this sort are included in the problems for this chapter.

4.9 Exercises

Section 4.1: Conservation of energy in 1-D

1. **Heading to zero** $*$

   A particle moves toward $x = 0$ under the influence of a potential $V(x) = -A|x|^n$ (assume $n > 0$). The particle has barely enough energy to reach $x = 0$. For what values of n will it reach $x = 0$ in a finite time?

2. **Heading to infinity** $*$

   A particle moves away from the origin under the influence of a potential $V(x) = -A|x|^n$. For what values of n will it reach infinity in a finite time?

3. **Work in different frames** $**$

   A mass m is initially at rest. A constant force F (directed to the right) acts on it over a distance d. The increase in kinetic energy is therefore Fd.

   Consider the situation from the point of view of someone moving to the left at speed V. Show explicitly that this person measures an increase in kinetic energy equal to force times distance.

4. **Beads on a hoop** $**$

   Two beads of mass m are positioned at the top of a frictionless hoop of mass M and radius R, which stands vertically on the ground. The beads are given tiny kicks, and they slide down the hoop, one to the right and one to the left. What is the smallest value of m/M for which the hoop will rise up off the ground, at some point during the motion?

Section 4.4: Gravity due to a sphere

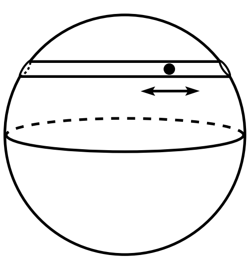


Figure 4.20

5. **Speedy travel** $**$

   A straight tube is drilled between two points on the earth, as shown in Fig. 4.20. An object is dropped into the tube. What is the resulting motion? How long does it take to reach the other end? You may ignore friction, and you may assume (erroneously) that the density of the earth in constant.

Section 4.5: Conservation of linear momentum

6. **Speedy rockets** $**$

   Assume that it is impossible to build a structurally sound container that can hold fuel of more than, say, nine times its mass. It would then seem like the limit for the speed of a rocket is $u \ln 10$.

   How can you make a rocket that goes faster than this?

7. **Maximum P and E of rocket** $*$

A rocket ejects its fuel at a constant rate per time. What is the mass of the rocket (including unused fuel) when its momentum is maximum? What is the mass when its energy is maximum?

Section 4.7: Collisions

8. **Maximum number of collisions** ∗∗

   N balls are constrained to move in one-dimension. What is the maximum number of collisions they may have among themselves? (Assume the collisions are elastic.)

9. **Triangular room** ∗∗

   A ball is thrown against a wall of a very long triangular room which has vertex angle θ. The initial direction of the ball is parallel to the angle bisector (see Fig. 4.21). How many bounces does the ball make?

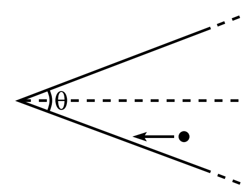


Figure 4.21

10. **Bouncing between rings** ∗∗

    Two circular rings, in contact with each other, stand in a vertical plane (see Fig. 4.22). Each has radius R. A small ball, with mass m and negligible size, bounces elastically back and forth between the rings. (Assume that the rings are held in place, so that they always remain in contact with each other.) Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola. Let this parabola hit the rings at an angle θ from the horizontal.

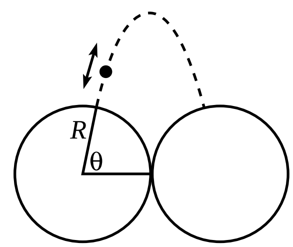


Figure 4.22

    (a) Let $\Delta P_x(\theta)$ be the magnitude of the change in the horizontal component of the ball's momentum, at each bounce. For what angle θ is $\Delta P_x(\theta)$ maximum?

    (b) Let S be the speed of the ball just before or after a bounce. And let $\overline{F}_x(\theta)$ be the average (over a long period of time) of the magnitude of the horizontal force needed to keep the rings in contact with each other (for example, the average tension in a rope holding the rings together). Consider the two limits: (1) $\theta \approx \epsilon$, and (2) $\theta \approx \pi/2 - \epsilon$, where ϵ is very small.

       i. Derive approximate formulas for S, in these two limits.

       ii. Derive approximate formulas for $\overline{F}_x(\theta)$, in these two limits.

       Which of these two limits requires a larger $\overline{F}_x$?

11. **Bouncing between surfaces** ∗∗

    Consider a more general case of the previous problem. Now let a ball bounce back and forth between a surface defined by $f(x)$ (for $x > 0$) and $f(-x)$ (for $x < 0$) (see Fig. 4.23). Again, assume that initial conditions have been set up so that the ball's motion forever lies in one parabola (the ball bounces back and forth between the contact points at $(x_0, f(x_0))$ and $(-x_0, f(x_0))$, for some x_0).

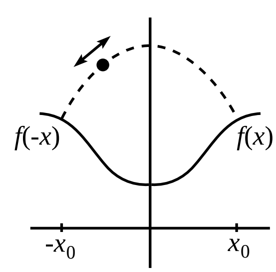


Figure 4.23

(a) Let $\Delta P_x(x_0)$ be the absolute value of the change in the horizontal component of the ball's momentum, at each bounce. For what function $f(x)$ is $\Delta P_x(x_0)$ independent of the contact position x_0?

(b) Let $\overline{F}_x(x_0)$ be the average of the magnitude of the horizontal force needed to keep the two halves of the surface together. For what function $f(x)$ is $\overline{F}_x(x_0)$ independent of the contact position x_0?

Section 4.8: Inherently inelastic processes

12. **Slowing down, speeding up** $*$

A plate of mass M initially moves horizontally at speed v on a frictionless table. A mass m is dropped vertically onto it and soon comes to rest with respect to the plate. How much energy is required to bring the system back up to speed v?

13. **Downhill dustpan** $***$

A dustpan slides down a plane inclined at angle θ. Dust is uniformly distributed on the plane, and the dustpan collects the dust in its path. After a long time, what is the acceleration of the dustpan? (Assume there is no friction between the dustpan and plane.)

4.10 Problems

Section 4.1: Conservation of energy in 1-D

1. **Exploding masses**

   A mass M moves with speed V. An explosion divides the mass in half, giving each half a speed v in the CM frame. Calculate the increase in kinetic energy in the lab frame. (Assume all motion is confined to one dimension.)

2. **Minimum length** $**$

   The shortest configuration of string joining three given points is the one shown in Fig. 4.24, where all three angles are 120°.[7]

   Devise an experimental proof of this fact by attaching three equal masses to three string ends, and then attaching the other three ends together (as shown in Fig. 4.24), and using whatever other props you need.

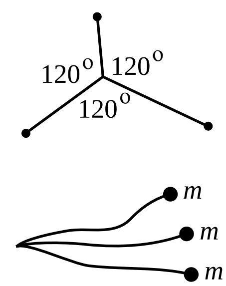


Figure 4.24

3. **Pulling the pucks** $**$

   (a) A string of length 2ℓ connects two hockey pucks which lie on frictionless ice. A constant horizontal force, F, is applied to the midpoint of the string, perpendicular to it (see Fig. 4.25). How much kinetic energy is lost when the pucks collide, assuming they stick together?

   (b) The answer you obtained above should be very clean and nice. Find the slick solution (assuming you solved the problem the "normal" way, above) that makes it transparent why the answer is so nice. (This is a neat problem. Try not to peek at the solution.)

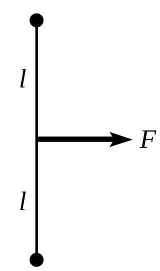


Figure 4.25

4. $V(x)$ **vs. a hill** $***$

   A ball, under the influence of gravity, slides along a surface whose height is given by the function $V(x)$ (see Fig. 4.26). What is the ball's horizontal acceleration, $\ddot{x}$?

   Is your answer the same as the $\ddot{x}$ for a particle moving in 1-D in the potential $mgV(x)$? If not, can you think of anything you can do with the surface to make the answer "yes"?

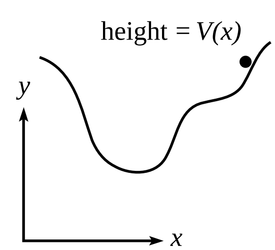


Figure 4.26

5. **Winding spring** $****$

   A mass m is attached to one end of a spring of zero equilibrium length, the other end of which is fixed. The spring constant is K. Initial conditions are set up so that the mass moves around in a circle of radius L on a frictionless horizontal table. (By "zero equilibrium length", we mean that the equilibrium length is negligible compared to L.)

   At a given time, a vertical pole (of radius a, with $a \ll L$) is placed in the ground next to the center of the circle, as shown. The spring winds around,

[7] If the three points form a triangle that has an angle greater than 120°, then the vertex of the string lies at the point where that angle is. We won't worry about this case.

and the mass eventually hits the pole. Assume that the pole is sticky, so that any part of the spring touching the pole does not slip. How long does it take the mass to hit the pole?

(Work in the approximation where $a \ll L$.)

Section 4.2: Small Oscillations

6. **Small oscillations**

   A particle moves under the influence of the potential $V(x) = -Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.

7. **More small oscillations**

   A particle moves under the influence of the potential $V(x) = A/x^2 - B/x$. Find the frequency of small oscillations around the equilibrium point.

8. **Hanging mass**

   A particle moves under the influence of the potential $V(x) = (k/2)x^2 + mgx$ (that is, it is a mass hanging from a spring). Find the frequency of small oscillations around the equilibrium point.

Section 4.4: Gravity due to a sphere

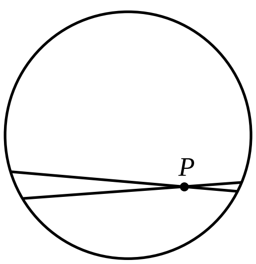


Figure 4.27

9. **Zero force inside a sphere** $**$

   Show that the gravitational force inside a spherical shell is zero by showing that the pieces of mass at the ends of the cones in Fig. 4.27 give canceling forces at point P.

10. **Escape velocity** $*$

    (a) Find the escape velocity (that is, the velocity above which a particle will escape to $r = \infty$) for a particle on a spherical planet of radius R and mass M. What is the numerical value for the earth? The moon? The sun?

    (b) Approximately how small must a spherical planet be in order for a human to be able to jump off? (Assume a density roughly equal to the earth's.)

11. **Ratio of potentials** $**$

    Find the ratio of the gravitational potential energy at the corner of a cube (of uniform mass density) to that at the center of the cube. (*Hint:* There's a slick way that doesn't involve any messy integrals.)

Section 4.5: Conservation of linear momentum

12. **Snowball** $*$

    A snowball is thrown against a wall. Where does its momentum go? Where does its energy go?

13. **Throwing at a car** ∗∗

For some odd reason, you decide to throw baseballs at a car (of mass M), which is free to move frictionlessly on the ground. You throw the balls at speed u, and at a mass rate of σ kg/s (assume the rate is continuous, for simplicity). If the car starts at rest, find its speed as a function of time, assuming that the balls bounce (elastically) directly backwards off the back window.

14. **Throwing at a car again** ∗∗

Do the previous problem, except now assume that the back window is open, so that the balls collect in the inside of the car.

15. **Chain on scale** ∗∗

A chain of length L and mass density σ is held such that it hangs vertically just above a scale. It is then released. What is the reading on the scale, as a function of the height of the top of the chain?

16. **Leaky bucket** ∗∗

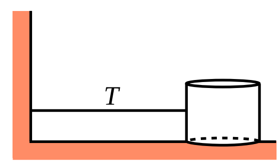


Figure 4.28

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). (See Fig. 4.28.) The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

The bucket is released. On the way to the wall, the bucket leaks sand at a rate $dm/dx = M/L$ (so the rate is constant with respect to distance, not time; note that dx is negative, so dm is also).

(a) What is the kinetic energy of the (sand in the) bucket, as a function of distance to the wall? What is its maximum value?

(b) What is the momentum of the bucket, as a function of distance to the wall? What is its maximum value?

17. **Another leaky bucket** ∗∗∗

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

The bucket is released. On the way to the wall, the bucket leaks sand at a rate $dm/dt = -bM$ (so the rate is constant with respect to time, not distance; we've factored out an 'M' here, just to make the calculations a little neater).

(a) Find $v(t)$ and $x(t)$ (for the times when the bucket contains a nonzero amount of sand).

(b) What is the maximum value of the bucket's kinetic energy?

(c) What is the maximum value of the magnitude of the bucket's momentum?

(d) For what value of b does the bucket become empty right when it hits the wall?

18. **Yet another leaky bucket** ***

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

The bucket is released. On the way to the wall, the bucket leaks sand at a rate proportional to its acceleration, that is, $dm/dt = b\ddot{x}$ (note that $\ddot{x}$ is negative, so dm is also).

(a) Find the mass as a function of time, $m(t)$.

(b) Find $v(t)$ and $x(t)$ (for the times when the bucket contains a nonzero amount of sand).
What is the speed right before all the sand leaves the bucket?

(c) What is the maximum value of the bucket's kinetic energy?

(d) What is the maximum value of the magnitude of the bucket's momentum?

(e) For what value of b does the bucket become empty right when it hits the wall?

Section 4.7: Collisions

19. **Right angle in billiards** *

A billiard ball collides elastically with an identical stationary one. Use the fact that the kinetic energy may be written as $m(\mathbf{v} \cdot \mathbf{v})/2$ to show that the angle between the resulting trajectories is $90°$.

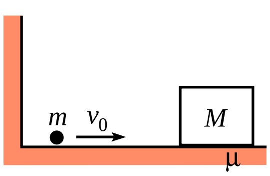


Figure 4.29

20. **Bouncing and recoiling** ***

A ball of mass m and initial speed v_0 bounces between a fixed wall and a block of mass M (with $M \gg m$). (See Fig. 4.29.) M is initially at rest. Assume the ball bounces elastically and instantaneously. The coefficient of kinetic friction between the block and the ground is μ. There is no friction between the ball and the ground.

What is the speed of the ball after the nth bounce off the block? How far does the block eventually move? How much total time does the block spend moving?

(Work in the approximation $M \gg m$, and assume that μ is large enough so that the block comes to rest by the time the next bounce occurs.)

21. **Drag force on a sheet** ***

A sheet of mass M moves with speed V through a region of space that contains particles of mass m and speed v. There are n of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume the particles do not interact with each other.

(a) Assuming $v \ll V$, what is the drag force per unit area on the sheet?

(b) Assuming $v \gg V$, what is the drag force per unit area on the sheet? (You may use the fact that the average speed of a particle in the x-direction is $v_x = v/\sqrt{3}$.)

22. **Drag force on a cylinder** **

A cylinder of mass M and radius R moves with speed V through a region of space that contains particles of mass m which are at rest. There are n of these particles per unit volume. The cylinder moves in a direction perpendicular to its axis. Assume $m \ll M$, and assume the particles do not interact with each other.

What is the drag force per unit length on the cylinder?

23. **Drag force on a sphere** **

A sphere of mass M and radius R moves with speed V through a region of space that contains particles of mass m which are at rest. There are n of these particles per unit volume. Assume $m \ll M$, and assume the particles do not interact with each other.

What is the drag force on the sphere?

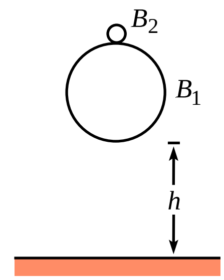


Figure 4.30

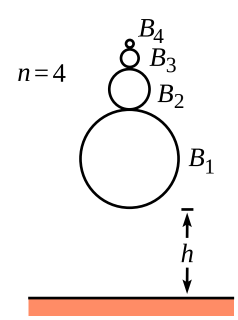


Figure 4.31

24. **Basketball and tennis ball** **

(a) A ball, B_2, with (very small) mass m_2 sits on top of another ball, B_1, with (very large) mass m_1 (see Fig. 4.30). The bottom of B_1 is at a height h above the ground, and the bottom of B_2 is at a height $h + d$ above the ground. The balls are dropped. To what height does the top ball bounce?

(You may work in the approximation where m_1 is much heavier than m_2. Assume that the balls bounce elastically. And assume, for the sake of having a nice clean problem, that the balls are initially separated by a small distance, and that the balls bounce instantaneously.)

(b) n balls, $B_1, \ldots, B_n$, having masses $m_1, m_2, \ldots, m_n$ (with $m_1 \gg m_2 \gg \cdots \gg m_n$), sit in a vertical stack (see Fig. 4.31). The bottom of B_1 is at a height h above the ground, and the bottom of B_n is at a height $h + \ell$ above the ground. The balls are dropped. In terms of n, to what height does the top ball bounce?

(Work in the approximation where m_1 is much heavier than m_2, which is much heavier than m_3, etc., and assume that the balls bounce elastically. Also, make the "nice clean problem" assumptions as in part (a).)

If $h = 1$ meter, what is the minimum number of balls needed in order for the top one to bounce to a height of at least 1 kilometer?

(Assume the balls still bounce elastically [which is not likely, in reality]. Ignore wind resistance, etc., and assume that ℓ is negligible here.)

Section 4.8: Inherently inelastic processes

25. **Colliding masses** $*$

A mass M, initially moving at speed v, collides and sticks to a mass m, initially at rest. Assume $M \gg m$. What are the final energies of the two masses, and how much energy is lost to heat, in:

(a) The lab frame?

(b) The frame in which M is initially at rest?

(Work in the approximation where $M \gg m$.)

26. **Pulling a chain** $**$

A chain of length L and mass density σ lies straight on a frictionless horizontal surface. You grab one end and pull it back along itself, in a parallel manner. Assume that you pull it at constant speed v. What force must you apply? What is the total work that you do, by the time the chain is straightened out? Is any energy lost to heat?

27. **Pulling a rope** $**$

A rope of length L and mass density σ lies in a heap on the floor. You grab an end and pull horizontally with constant force F. What is the position of the end of the rope, as a function of time (while it is unraveling)?

28. **Falling rope** $***$

(a) A rope of length L lies in a straight line on a frictionless table, except for a very small piece of it which hangs down through a hole in the table. The rope is released, and it slides down through the hole. What is the speed of the rope at the instant it loses contact with the table?

(b) A rope of length L lies in a heap on a table, except for a very small piece of it which hangs down through a hole in the table. The rope is released, and it unravels and slides down through the hole. What is the speed of the rope at the instant it loses contact with the table? (Assume the rope is greased, so that it has no friction with itself.)

29. **Raising the rope** $**$

A rope of length L and mass density σ lies in a heap on the floor. You grab one end of the rope and pull upward with a force such that the rope moves at constant speed v.

What is the total work you do, by the time the rope is completely off the floor? How much energy is lost to heat, if any?

30. **The raindrop** ****

Assume that a cloud consists of tiny water droplets suspended in air (uniformly distributed, and at rest), and consider a raindrop falling through them. After a long time, what is the acceleration of the raindrop?

(Assume that when the raindrop hits a water droplet, the droplet's water gets added to the raindrop. Also, assume that the raindrop is spherical at all times. Ignore air resistance on the raindrop.)

4.11 Solutions

1. **Exploding masses**

   First Solution: In the CM frame, the increase in kinetic energy is

$$2\frac{1}{2}(M/2)v^2 = \frac{1}{2}Mv^2. \tag{4.82}$$

   This increase is due to the work done by the explosion, which is the same in any frame. So $Mv^2/2$ is the increase in kinetic energy in the lab frame, too.

   Using this reasoning, it is clear that $Mv^2/2$ is the answer even if the two pieces aren't constrained to move along the direction of the initial velocity.

   Second Solution: We can also do this problem in the lab frame. The final velocities in the lab frame are $V + v$ and $V - v$. The total kinetic energy is therefore

$$\frac{1}{2}(M/2)(V+v)^2 + \frac{1}{2}(M/2)(V-v)^2 = \frac{1}{2}(M/2)2(V^2+v^2). \tag{4.83}$$

   The initial KE was $MV^2/2$, so the increase is $Mv^2/2$.

   Remark: From the first solution, we know that $Mv^2/2$ is the answer even if the two pieces aren't constrained to move along the direction of the initial velocity. Let's check this explicitly. Let the x-y plane be the plane containing the initial and final velocities. Let the initial velocity lie along the x axis. If the new velocities make an angle θ with respect to the x-axis in the CM frame, then the final velocities in the lab frame are

$$\begin{aligned} \mathbf{V}_1 &= (V + v\cos\theta, v\sin\theta), \\ \mathbf{V}_2 &= (V - v\cos\theta, -v\sin\theta). \end{aligned} \tag{4.84}$$

   The total kinetic energy is therefore

$$\frac{1}{2}(M/2)V_1^2 + \frac{1}{2}(M/2)V_2^2 = \frac{1}{2}(M/2)2(V^2+v^2). \tag{4.85}$$

   The initial KE was $MV^2/2$, so the increase is $Mv^2/2$. ♣

2. **Minimum length**

   Cut three holes in a table, which represent the three given points. Drop the masses through the holes, and let the system reach its equilibrium position.

   The equilibrium position is the one with the lowest potential energy of the masses, i.e., the one with the most string hanging below the table, i.e., the one with the least string lying on the table. So this is the desired 'minimum length' configuration.

   What are the angles at the vertex of the string? The tensions in all three strings are mg. The (massless) vertex of the string is in equilibrium, so the net force on it must be zero. Therefore, each string must bisect the angle formed by the other two. This implies that the three strings must have 120° angles between them.

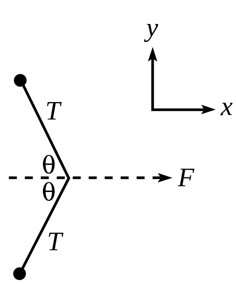


Figure 4.32

3. **Pulling the pucks**

   (a) Let θ be the angle shown in Fig. 4.32. The tension in the string is then $F/(2\cos\theta)$, because the force on the (massless) kink in the string must be zero. Consider the 'top' puck. The force in the y-direction is $-F\tan\theta/2$. The work done on this puck in the y-direction is therefore

$$\begin{aligned} W_y &= \int_\ell^0 \frac{-F\tan\theta}{2} dy \\ &= \int_{\pi/2}^0 \frac{-F\tan\theta}{2} d(\ell\sin\theta) \\ &= \int_{\pi/2}^0 \frac{-F\ell\sin\theta}{2}\, d\theta \\ &= \frac{F\ell\cos\theta}{2}\Big|_{\pi/2}^0 \\ &= \frac{F\ell}{2}. \end{aligned} \tag{4.86}$$

The kinetic energy lost when the pucks stick together is twice this. Therefore,

$$\text{KE}_{\text{loss}} = F\ell. \tag{4.87}$$

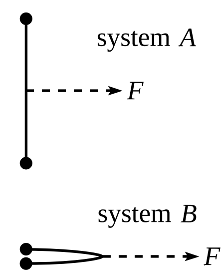


Figure 4.33

(b) Consider two systems, A and B (see Fig. 4.33). A is the original setup, while B starts with θ already at zero. Let the pucks in both systems start at $x = 0$. As the force F is applied, all the pucks will have the same $x(t)$, because the same force in the x-direction, $F/2$, is being applied to every puck at all times. After the collision, both systems look exactly the same.

Let the collision of the pucks occur at $x = d$. At this point, Fd work has been done on system A, while only $F(d - \ell)$ work has been done on system B. Since both systems have the same KE after the collision, the extra $F\ell$ of work done on system A must be what is lost in the collision.

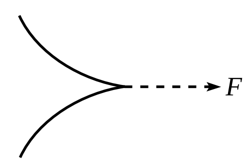


Figure 4.34

REMARK: The reasoning in this second solution makes it clear that the same $F\ell$ result holds even if we have many masses, or if we have rope with a continuous mass distribution (so that the rope flops down, as in Fig. 4.34). The only requirement is that the mass be symmetrically distributed around the midpoint. Analyzing this more general setup along the lines of the first solution above would be extremely tedious, at best. ♣

4. $V(x)$ **vs. a hill**

Incorrect Solution: Here is a common incorrect way to do the problem. See if you can spot the error.

The slope of the curve is $\tan\theta \equiv V'(x)$, so the acceleration along the curve is $g\sin\theta = g(V'/\sqrt{1+V'^2})$, to the left. Taking the horizontal component of this acceleration brings in a factor of $\cos\theta = 1/\sqrt{1+V'^2}$. So we have

$$\ddot{x} = \frac{-gV'}{1+V'^2}. \tag{4.88}$$

On the other hand, the acceleration of a particle moving in the potential $mgV(x)$ is

$$\ddot{x} = \frac{F}{m} = \frac{1}{m}\left(\frac{-d(mgV)}{dx}\right) = -gV'. \tag{4.89}$$

This equation is not the same as eq. (4.88) (unless $1 + V'^2 = 1$, that is, $V' = 0$ everywhere, in which case the motion is trivial). Therefore, the two given setups give different functions $x(t)$.

This conclusion is indeed correct. However, eq. (4.88) is *not* correct, as we will see below. It is via incorrect reasoning that we obtain the correct answer that the setups are different.

Before doing the problem correctly, let us note one (invalid) consequence of (the incorrect) eq. (4.88). It looks like we can mimic a potential $mgV(x)$ be choosing a surface whose height, $H(x)$, satisfies

$$\frac{-gH'}{1+H'^2} = -gV' \qquad \Longrightarrow \qquad H' = \frac{1 \pm \sqrt{1-4V'^2}}{2V'}. \tag{4.90}$$

Given a $V(x)$, it seems like all we have to do is integrate this (perhaps numerically) to find $H(x)$. The one condition for H to be real is $|V'(x)| \le 1/2$, for all x. One example is $V(x) = (1/2)\ln(1+x^2)$ and $H(x) = x^2/2$. However, this is all garbage, as we will now show.

Correct Solution: The error in deriving eq. (4.88) was that we forgot to take into account the possible change in the curve's slope. (Eq. (4.88) is true for straight lines.) We addressed only the acceleration due to a change in *speed*. We forgot about acceleration due to a change in the *direction* of motion. Here's how to do the problem properly.

In terms of the horizontal speed, $\dot{x}$, the speed along the curve is $v = \dot{x}/\cos\theta = \dot{x}\sqrt{1+V'^2}$. (Dots denote d/dt. Primes denote d/dx.) We know that $dv/dt = -g\sin\theta = -g(V'/\sqrt{1+V'^2})$. Therefore,

$$\begin{aligned}\frac{-gV'}{\sqrt{1+V'^2}} = \frac{dv}{dt} &= \frac{d}{dt}\left(\dot{x}\sqrt{1+V'^2}\right)\\ &= \ddot{x}\sqrt{1+V'^2} + \frac{\dot{x}V'\frac{dV'}{dt}}{\sqrt{1+V'^2}}.\end{aligned} \tag{4.91}$$

If the second term were not here, we would recover eq. (4.88). But alas, the correct statement is

$$\ddot{x} = \frac{-gV'}{1+V'^2} - \frac{\dot{x}V'\frac{dV'}{dt}}{1+V'^2}. \tag{4.92}$$

This doesn't look much like the $-gV'$ in eq. (4.89). We can put it in a much nicer form by noting that

$$\begin{aligned}\dot{x}V'\frac{dV'}{dt} &= \dot{x}V'\frac{d}{dt}\left(\frac{dV/dt}{dx/dt}\right)\\ &= \dot{x}V'\left(\frac{\dot{x}\ddot{V} - \dot{V}\ddot{x}}{\dot{x}^2}\right)\\ &= V'\ddot{V} - V'\ddot{x}\left(\frac{\dot{V}}{\dot{x}}\right)\\ &= V'\ddot{V} - V'^2\ddot{x}.\end{aligned} \tag{4.93}$$

Substituting this into eq. (4.92) gives the rather aesthetically pleasing result,

$$\ddot{x} = -(g+\ddot{V})V'. \tag{4.94}$$

Therefore, if the rate of change of the particle's vertical speed (that is, $\ddot{V}$) is not uniform (as is invariably the case), then this does not equal the $-gV'$ in eq. (4.89).

There is generally no way to construct a curve with height $H(x)$ which gives the same motion as a potential $V(x)$, for all initial conditions. We would need $(g+\ddot{H})H' = V'$, for all x. But at a given x, the quantities V' and H' are fixed, whereas $\ddot{H}$ depends on the initial conditions.

Eq. (4.94) holds the key to constructing a situation that does mimic a 1-D potential $V(x)$. All we have to do is get rid of the $\ddot{V}$ term. So here's what you do: Build a rigid curve with height given by $V(x)$, put a ball on it, and then move the curve up and down in such a manner that the ball stays at the same level w.r.t. the ground (or moves with constant vertical speed). This will make the $\ddot{V}$ term vanish, as desired. (And note that the vertical movement of the curve doesn't change V').

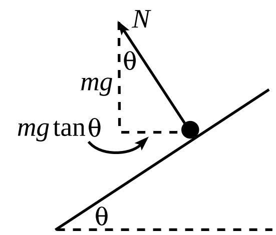


Figure 4.35

REMARK: Now that we've gone through all of the above calculations, here's a much simpler way to see what's going on. Things are much clearer if we examine the normal force, N, acting on the ball (see Fig. 4.35). The upward component is $N\cos\theta$. If we demand that the ball not accelerate in the y-direction, then we must have $N\cos\theta = mg$. Therefore, the horizontal force acting on the ball is $-N\sin\theta = -mg\tan\theta$. But $\tan\theta = V'(x)$. So the horizontal force is $-mgV'$, as desired. This line of reasoning also makes it clear that eq. (4.94) is correct for arbitrary $\ddot{V}$. ♣

5. **Winding spring**

Let $\theta(t)$ be the angle through which the spring moves. Let $x(t)$ be the length of the unwrapped part of the spring. Let $v(t)$ be the speed of the mass. And let $k(t)$ be the spring constant of the unwrapped part of the spring. (The manner in which k changes will be derived below.)

Using the approximation $a \ll L$, we may say that the mass undergoes approximate circular motion. (This approximation will break down when x becomes of order a, but the time during which this is true is negligible compared to the total time. We will justify this at the end of the solution.) The instantaneous center of the circle is the point where the spring touches the pole. $F = ma$ along the instantaneous radial direction gives

$$\frac{mv^2}{x} = kx. \tag{4.95}$$

Using this value of v, the frequency of the circular motion is given by

$$\omega \equiv \frac{d\theta}{dt} = \frac{v}{x} = \sqrt{\frac{k}{m}}. \tag{4.96}$$

The spring constant, $k(t)$, of the unwrapped part of the spring is inversely proportional to its equilibrium length. (For example, if you cut a spring in half, the resulting springs have twice the original spring constant). All equilibrium lengths in this problem are infinitesimally small (compared to L), but the inverse relation between k and equilibrium length still holds. (If you want, you can think of the equilibrium length as a measure of the total number of spring atoms that remain in the unwrapped part.)

Note that the change in angle of the contact point on the pole equals the change in angle of the mass around the pole (which is θ.) Consider a small interval of time during which the unwrapped part of the spring stretches a small amount and moves through an angle $d\theta$. Then a length $a\,d\theta$ becomes wrapped on the pole. So the fractional decrease in the equilibrium length of the unwrapped part is (to first order in $d\theta$) equal to $(a\,d\theta)/x$. From the above paragraph, the new spring constant is therefore

$$k_{\text{new}} = \frac{k_{\text{old}}}{1-\frac{a\,d\theta}{x}} \approx k_{\text{old}}\left(1+\frac{a\,d\theta}{x}\right). \tag{4.97}$$

Therefore, $dk = ka\,d\theta/x$. Dividing by dt gives

$$\dot{k} = \frac{ka\omega}{x}. \tag{4.98}$$

The final equation we need is the one for energy conservation. At a given instant, consider the sum of the kinetic energy of the mass, and the potential energy of the unwrapped part of the spring. At a time dt later, a tiny bit of this energy will be stored in the newly-wrapped little piece. Letting primes denote quantities at this later time, conservation of energy gives

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k'x'^2 + \frac{1}{2}m'v'^2 + \left(\frac{1}{2}kx^2\right)\left(\frac{a\,d\theta}{x}\right). \tag{4.99}$$

The last term is (to lowest order in $d\theta$) the energy stored in the newly-wrapped part, because $a\,d\theta$ is its length. Using eq. (4.95) to write the v's in terms of the x's, this becomes

$$kx^2 = k'x'^2 + \frac{1}{2}kxa\,d\theta. \tag{4.100}$$

In other words, $-(1/2)kxa\,d\theta = d(kx^2)$. Dividing by dt gives

$$\begin{aligned} -\frac{1}{2}kxa\omega &= \frac{d(kx^2)}{dt} \\ &= \dot{k}x^2 + 2kx\dot{x} \\ &= \left(\frac{ka\omega}{x}\right)x^2 + 2kx\dot{x}, \end{aligned} \tag{4.101}$$

where we have used eq. (4.98). Therefore,

$$\dot{x} = -\frac{3}{4}a\omega. \tag{4.102}$$

We must now solve the two couple differential equations, eqs. (4.98) and (4.102). Dividing the latter by the former gives

$$\frac{\dot{x}}{x} = -\frac{3}{4}\frac{\dot{k}}{k}. \tag{4.103}$$

Integrating and exponentiating gives

$$k = \frac{KL^{4/3}}{x^{4/3}}, \tag{4.104}$$

where the numerator is obtained from the initial conditions, $k = K$ and $x = L$. Plugging eq. (4.104) into eq. (4.102), and using $\omega = \sqrt{k/m}$, gives

$$x^{2/3}\dot{x} = -\frac{3aK^{1/2}L^{2/3}}{4m^{1/2}}. \tag{4.105}$$

Integrating, and using the initial condition $x = L$, gives

$$x^{5/3} = L^{5/3} - \left(\frac{5aK^{1/2}L^{2/3}}{4m^{1/2}}\right)t. \tag{4.106}$$

So, finally,

$$x(t) = L\left(1 - \frac{t}{T}\right)^{3/5}, \tag{4.107}$$

where

$$T = \frac{4}{5}\frac{L}{a}\sqrt{\frac{m}{K}} \tag{4.108}$$

is the time for which $x(t) = 0$ and the mass hits the pole.

REMARKS:

(a) Note that the angular momentum of the mass around the center of the pole is *not* conserved in this problem, because the force is not a central force.

(b) Integrating eq. (4.102) up to the point when the mass hit the pole gives $-L = -(3/4)a\theta$. But $a\theta$ is the total length wrapped around the pole, which we see is equal to $4L/3$.

(c) We may now justify the assumption of approximate circular motion used above. The exact radial $F = ma$ equation of motion is $-kx = m\ddot{x} - mv^2/x$. Therefore, our neglect of the $\ddot{x}$ term in eq. (4.95) is valid as long as $kx \gg m|\ddot{x}|$. Using our result for $x(t)$ in eq. (4.107), this condition is

$$kL\left(1 - \frac{t}{T}\right)^{3/5} \gg \frac{6}{25}\frac{mL}{T^2}\left(1 - \frac{t}{T}\right)^{-7/5}. \tag{4.109}$$

Using the above expression for T, this becomes

$$kL\left(1 - \frac{t}{T}\right)^{3/5} \gg \frac{3}{8}\frac{Ka^2}{L}\left(1 - \frac{t}{T}\right)^{-7/5}. \tag{4.110}$$

We can write k as a function of t if we wish, but there is no need, since $k \geq K$. Eq. (4.110) is valid up to t arbitrarily close to T, as long as a is sufficiently small compared to L. The solution for x in eq. (4.107) is therefore an approximate solution to the true equation of motion. ♣

6. **Small oscillations**

We will calculate the equilibrium point x_0, and then use $\omega = \sqrt{V''(x_0)/m}$. We have

$$V'(x) = -Ce^{-ax}x^{n-1}(n - ax). \tag{4.111}$$

So $V'(x) = 0$ when $x = n/a \equiv x_0$. The second derivative is

$$V''(x) = -Ce^{-ax}x^{n-2}\Big((n - 1 - ax)(n - ax) - ax\Big). \tag{4.112}$$

Plugging in $x_0 = n/a$ simplifies this a bit, and we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{Ce^{-n}n^{n-1}}{ma^{n-2}}}. \tag{4.113}$$

7. **More small oscillations**

We will calculate the equilibrium point x_0, and then use $\omega = \sqrt{V''(x_0)/m}$. We have

$$V'(x) = -\frac{2A}{x^3} + \frac{B}{x^2}. \tag{4.114}$$

So $V'(x) = 0$ when $x = 2A/B \equiv x_0$. The second derivative is

$$V''(x) = \frac{6A}{x^4} - \frac{2B}{x^3}. \tag{4.115}$$

Plugging in $x_0 = 2A/B$, we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{B^4}{8mA^3}}. \tag{4.116}$$

8. **Hanging mass**

We will calculate the equilibrium point x_0, and then use $\omega = \sqrt{V''(x_0)/m}$. We have

$$V'(x) = kx + mg. \tag{4.117}$$

So $V'(x) = 0$ when $x = -mg/k \equiv x_0$. The second derivative is

$$V''(x) = k. \tag{4.118}$$

So we have

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{k}{m}}. \tag{4.119}$$

This is independent of x_0, which is what we expect. The only effect of gravity is to change the equilibrium position. If x_r is the position relative to x_0, then the force is $-kx_r$, so it still looks like a regular spring. (This works, of course, only because the spring force in linear.)

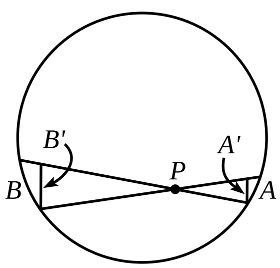


Figure 4.36

9. **Zero force inside a sphere**

Let a be the distance from P to piece A, and let b be the distance from P to piece B (see Fig. 4.36). Draw the 'perpendicular' bases of the cones; call them A' and B'. The ratio of the areas of A' and B' is a^2/b^2.

The key point to realize here is that the angle between the planes of A and A' is the same as that between B and B'. (This is true because the chord between A and B meets the circle at equal angles at its ends. A circle is the only figure with this property for all chords.) So the ratio of the areas of A and B is also a^2/b^2. But the gravitational force decreases like $1/r^2$, so the forces at P due to A and B are equal in magnitude (and opposite in direction, of course).

10. **Escape velocity**

(a) The initial kinetic energy, $mv_{\text{esc}}^2/2$, must account for the gain in potential energy, GMm/R, out to infinity. Therefore,

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}. \tag{4.120}$$

In terms of the acceleration, $g = GM/R^2$, at the surface of the earth, v_{esc} may be written as $v_{\text{esc}} = \sqrt{2gR}$. Using $M = 4\pi\rho R^3/3$, we may also write it as $v_{\text{esc}} = \sqrt{8\pi G R^2\rho/3}$.

We'll use the values of g in Appendix I to find the numerical values:

For the earth, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(10\,\text{m/s}^2)(6.4\cdot 10^6\,\text{m})} \approx 11,300\,\text{m/s}$.

For the moon, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(1.62\,\text{m/s}^2)(1.74\cdot 10^6\,\text{m})} \approx 2,370\,\text{m/s}$.

For the sun, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(274\,\text{m/s}^2)(7\cdot 10^8\,\text{m})} \approx 620,000\,\text{m/s}$.

REMARK: Another reasonable question to ask is: what is the escape velocity from the sun for an object located where the earth is? (But imagine that the earth isn't there.) The answer is $\sqrt{2GM_S/R_{E,S}}$, where $R_{E,S}$ is the earth-sun distance. Numerically, this is $\sqrt{2(6.67\cdot 10^{-11})(2\cdot 10^{30})/(1.5\cdot 10^{11})} \approx 42,200\,\text{m/s}$. ♣

(b) To get a rough answer, let's assume that the initial speed of the jump is the same on the small planet as it is on the earth (this probably isn't quite true, but it's close enough for the purposes here). A good jump on the earth is one meter. For this jump, $mv^2/2 = mg(1)$. Therefore, $v = \sqrt{2g} \approx \sqrt{20}$m/s. So we want $\sqrt{20} = \sqrt{8\pi G R^2 \rho/3}$. Using $\rho \approx 5500\,\text{kg/m}^3$, we find $R \approx 2.5\,\text{km}$. On such a planet, you should indeed tread lightly. And don't sneeze.

11. **Ratio of potentials**

Let ρ be the mass density of the cube. Let V_ℓ^{cor} be the potential energy at the corner of a cube of side ℓ. Let V_ℓ^{cen} be the potential energy at the center of a cube of side ℓ. By dimensional analysis,

$$V_\ell^{\text{cor}} \propto \frac{Q}{\ell} = \rho \ell^2. \tag{4.121}$$

Therefore,[8]

$$V_\ell^{\text{cor}} = 4V_{\ell/2}^{\text{cor}}\,. \tag{4.122}$$

But by superposition, we have

$$V_\ell^{\text{cen}} = 8V_{\ell/2}^{\text{cor}}\,, \tag{4.123}$$

because the center of the larger cube lies at a corner of the eight smaller cubes of which it is made. Therefore,

$$\frac{V_\ell^{\text{cor}}}{V_\ell^{\text{cen}}} = \frac{4V_{\ell/2}^{\text{cor}}}{8V_{\ell/2}^{\text{cor}}} = \frac{1}{2}\,. \tag{4.124}$$

12. **Snowball**

All of the momentum of the snowball, mv, goes into the earth. So the earth translates (and rotates) a tiny bit faster (or slower, depending on which way the snowball was thrown).

Let M be the mass of the earth. Let V be the final speed of the earth (with respect to the original rest frame of the earth). Then $V \approx (m/M)v$, with $m \ll M$. The kinetic energy of the earth is therefore

$$\frac{1}{2}M\left(\frac{mv}{M}\right)^2 = \frac{1}{2}mv^2\left(\frac{m}{M}\right) \ll \frac{1}{2}mv^2. \tag{4.125}$$

(There is also a rotational kinetic-energy term of the same order of magnitude; but this doesn't matter.) So essentially none of the snowball's energy goes into the earth. It therefore all goes into the form of heat, which melts some of the snow.

This is a general result for a small object hitting a large object: The large object picks up all the momentum but essentially none of the energy.

13. **Throwing at a car** $**$

Let the speed of the car be $v(t)$. Consider the collision of a given ball (let it have mass ϵ) with the car. In the instantaneous rest frame of the car, the speed of the ball is $u - v$. In this frame, the ball reverses velocity when it bounces, so its change in momentum is (negative) $2\epsilon(u - v)$. The change in momentum is the same in the lab frame (since the two frames are related by a given speed at any instant). Therefore,

[8] In other words, imagine expanding a cube with side $\ell/2$ to one with side ℓ. If we consider corresponding pieces of the two cubes, then the larger piece has $2^3 = 8$ times the mass of the smaller. But corresponding distances are twice as big in the large cube as in the small cube. Therefore, the larger piece contributes $8/2 = 4$ times as much to V_ℓ^{cor} as the smaller piece contributes to $V_{\ell/2}^{\text{cor}}$.

in the lab frame the car gains a momentum of $2\epsilon(u-v)$, for each ball that hits it. The rate of change in momentum of the car (that is, the force) is thus

$$M\,dv/dt = dp/dt = 2\sigma'(u-v), \tag{4.126}$$

where σ' is the rate at which mass hits the car. σ' is related to the given σ by $\sigma' = \sigma(u-v)/u$, because although you throw the balls at speed u, the relative speed of the balls and the car is only $(u-v)$. We therefore arrive at

$$\begin{aligned}
M\frac{dv}{dt} &= \frac{2(u-v)^2\sigma}{u} \\
\Longrightarrow \quad \int_0^v \frac{dv}{(u-v)^2} &= \int_0^t \frac{2\sigma}{Mu}\,dt \\
\Longrightarrow \quad \frac{1}{u-v} - \frac{1}{u} &= \frac{2\sigma t}{Mu} \\
\Longrightarrow \quad v(t) &= \frac{\left(\frac{2\sigma t}{M}\right)u}{1+\frac{2\sigma t}{M}}.
\end{aligned} \tag{4.127}$$

REMARK: The speed $v(t)$ may be integrated to obtain $x(t) = ut - u(M/2\sigma)\ln(1+2\sigma t/M)$. Therefore, even though its speed approaches u, the car will eventually be an arbitrarily large distance behind a ball with constant speed u (for example, pretend that your first ball missed the car and continued to travel forward at speed u). ♣

14. **Throwing at a car again** $**$

We'll carry over some results from the solution for the previous problem. The only change in the calculation of the force on the car is that since the balls don't bounce backwards, we don't pick up the factor of 2 in eq. (4.126). The force is therefore

$$m\frac{dv}{dt} = \frac{(u-v)^2\sigma}{u}, \tag{4.128}$$

where $m(t)$ is the mass of the car-plus-contents, as a function of time. The main difference between this problem and the previous one is that this mass m changes, because the balls are collecting inside the car. As in the previous problem, the rate at which the balls enter the car is $\sigma' = \sigma(u-v)/u$. Therefore,

$$\frac{dm}{dt} = \frac{(u-v)\sigma}{u}. \tag{4.129}$$

We must now solve the two preceding differential equations. Dividing eq. (4.128) by eq. (4.129), and separating variables, gives

$$\int_0^v \frac{dv}{u-v} = \int_M^m \frac{dm}{m} \quad \Longrightarrow \quad m = \frac{Mu}{u-v}. \tag{4.130}$$

(Note that $m \to \infty$ as $v \to u$, as it should.) Substituting this value for m into either of eqs. (4.128) or (4.129) gives

$$\begin{aligned}
\int_0^v \frac{dv}{(u-v)^3} &= \int_0^t \frac{\sigma}{Mu^2}\,dt \\
\Longrightarrow \quad \frac{1}{2(u-v)^2} - \frac{1}{2u^2} &= \frac{\sigma t}{Mu^2} \\
\Longrightarrow \quad v(t) &= u\left(1 - \frac{1}{\sqrt{1+\frac{2\sigma t}{M}}}\right).
\end{aligned} \tag{4.131}$$

REMARK: Note that if we erase the square-root sign here, we obtain the answer to the previous problem, eq. (4.127). For a given t, the $v(t)$ in eq. (4.131) is smaller than the $v(t)$ in eq. (4.127). It is clear that this should be the case, since the balls have less of an effect on $v(t)$, because (a) they now don't bounce back, and (b) the mass of the car-plus-contents is now larger. ♣

15. **Chain on scale**

Let y be the height of the chain. Let F be the desired force applied by the scale. The net force on the chain is $F - (\sigma L)g$ (with upward taken to be positive). The momentum of the chain is $(\sigma y)\dot{y}$. Equating the net force to the change in momentum gives

$$\begin{aligned} F - \sigma L g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{4.132}$$

The part of the chain that is still above the scale is in free-fall. Therefore, $\ddot{y} = -g$, and $\dot{y} = \sqrt{2g(L-y)}$, the usual result for a falling object. Putting these into eq. (4.132) gives

$$\begin{aligned} F &= \sigma L g - \sigma y g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4.133}$$

This answer has the expected property of equaling zero when $y = L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading will of course simply be the weight of the chain, namely $(\sigma L)g$.

16. **Leaky bucket**

(a) **First Solution:** The initial position is $x = L$. The given rate of leaking implies that the mass of the bucket at position x is $m = M(x/L)$. Therefore, $F = ma$ gives

$$-T = \frac{Mx}{L}\ddot{x} \qquad \Longrightarrow \qquad \frac{TL}{M} = -xv\frac{dv}{dx}. \tag{4.134}$$

Separating variables and integrating yields $C - (TL/M)\ln x = v^2/2$, which we may write as

$$B - \frac{TL}{M}\ln(x/L) = \frac{v^2}{2}. \tag{4.135}$$

(since it's much nicer to have dimensionless arguments in a log). The integration constant, B, must be 0, because $v = 0$ when $x = L$.

The kinetic energy at position x is therefore

$$E = \frac{mv^2}{2} = \left(\frac{Mx}{L}\right)\frac{v^2}{2} = -Tx\ln(x/L). \tag{4.136}$$

In terms of the fraction $z \equiv x/L$, we have $E = -TLz\ln z$. Setting $dE/dz = 0$ to find the maximum gives

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{TL}{e}. \tag{4.137}$$

Note that $E_{\max}$ is independent of M. (This problem was just an excuse to give you an exercise where the answer contains an "e".)

REMARK: We started this solution off by writing down $F = ma$ (where m is the mass of the bucket), and you may be wondering why we didn't use $F = dp/dt$ (where p is the momentum of the bucket). These are clearly different, because $dp/dt = d(mv)/dt = ma + (dm/dt)v$. We used $F = ma$, because at any instant, the mass m is what is being accelerated by the force F.

It is indeed true that $F = dp/dt$, if you let F be *total* force in the problem, and let p be the *total* momentum. The tension T is the only force in the problem, since we've assumed the ground to be frictionless. However, the total momentum consists of both the sand in the bucket and the sand that has leaked out and is sliding along the ground.[9] If you use $F = dp/dt$, with p being the total momentum, then you simply arrive at $F = ma$, as you should check.

See Appendix E for further discussion on the use of $F = ma$ and $F = dp/dt$.

♣

Second solution: Consider a small interval during which the bucket moves from x to $x + dx$ (where dx is negative). The bucket's kinetic energy changes by $(-T)dx$ (this is a positive quantity) due to the work done by the spring, and also changes by a fraction dx/x (this is a negative quantity) due to the leaking. Therefore, $dE = -T\,dx + E\,dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}. \tag{4.138}$$

In solving this differential equation, it is convenient to introduce the variable $y = E/x$. Then $E' = xy' + y$. So eq. (4.138) becomes $xy' = -T$, or

$$dy = \frac{-T dx}{x}. \tag{4.139}$$

Integrating gives $y = -T \ln x + C$, which we may write as

$$y = -T \ln(x/L) + B \tag{4.140}$$

(to have a dimensionless argument in the log). Writing this in terms of E, we find

$$E = -Tx \ln(x/L), \tag{4.141}$$

as in the first solution. The integration constant, B, must be 0, because $E = 0$ when $x = L$.

(b) From eq. (4.135) (with $B = 0$), we have $v = \sqrt{2TL/M}\,\sqrt{-\ln z}$ (where $z \equiv x/L$). Therefore,

$$p = mv = (Mz)v = \sqrt{2TLM}\sqrt{-z^2 \ln z}. \tag{4.142}$$

Setting $dp/dz = 0$ to find the maximum gives

$$z = \frac{1}{\sqrt{e}} \qquad \Longrightarrow \qquad p_{\max} = \sqrt{\frac{TLM}{e}}. \tag{4.143}$$

We see that the location of $p_{\max}$ is independent of M,T,L, but its value is not.

[9] If the ground had friction, we would have to worry about its effect on both the bucket and the sand outside, if we wanted to use $F = dp/dt$, where p is the total momentum.

REMARK: $E_{\rm max}$ occurs at a later time (that is, closer to the wall) than $p_{\rm max}$ does. This is because v matters more in $E = mv^2/2$ than it does in $p = mv$. As far as E is concerned, it is beneficial for the bucket to lose a little more mass if it means being able to pick up a little more speed (up to a certain point). ♣

17. **Another leaky bucket**

(a) The given rate of leaking implies that the mass of the bucket at time t is $M(1-bt)$, for $t < 1/b$. Therefore, $F = ma$ gives

$$-T = M(1-bt)\ddot{x} \qquad \Longrightarrow \qquad \frac{-T\,dt}{M(1-bt)} = dv. \tag{4.144}$$

Integration yields

$$v(t) = \frac{T}{bM}\ln(1-bt), \tag{4.145}$$

where the constant of integration is 0, because $v = 0$ when $t = 0$. This equation is valid for $t < 1/b$ (provided that the bucket hasn't hit the wall yet, of course). Integrating $v(t)$ to get $x(t)$ gives (using $\int \ln y = y\ln y - y$)

$$x(t) = L - \frac{T}{b^2M} - \frac{T}{b^2M}\Big((1-bt)\ln(1-bt) - (1-bt)\Big), \tag{4.146}$$

where the constant of integration has been chosen so that $x = L$ when $t = 0$.

(b) The mass at time t is $M(1-bt)$. Using eq. (4.145), the kinetic energy at time t is (with $z \equiv 1 - bt$)

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(Mz)v^2 = \frac{T^2}{2b^2M}z\ln^2 z. \tag{4.147}$$

Taking the derivative to find the maximum, we obtain

$$z = \frac{1}{e^2} \qquad \Longrightarrow \qquad E_{\rm max} = \frac{2T^2}{e^2b^2M}. \tag{4.148}$$

(c) The mass at time t is $M(1-bt)$. Using eq. (4.145), the momentum at time t is (with $z \equiv 1 - bt$)

$$p = mv = (Mz)v = \frac{T}{b}z\ln z. \tag{4.149}$$

Taking the derivative to find the maximum magnitude, we obtain

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad |p|_{\rm max} = \frac{T}{eb}. \tag{4.150}$$

(d) We want $M(1-bt)$ to become zero right when x reaches 0. So we want $x = 0$ when $t = 1/b$. Eq. (4.146) then gives

$$0 = L - \frac{T}{b^2M} \qquad \Longrightarrow \qquad b = \sqrt{\frac{T}{ML}}. \tag{4.151}$$

REMARK: This is the only combination of M,T,L that has units of t^{-1}. But we needed to do the calculation to show that the numerical factor is 1. b clearly should increase with T and decrease with L. The dependence on M is not as obvious (although if b increases with T then it must decrease with M, from dimensional analysis). ♣

18. **Yet another leaky bucket**

(a) $F = ma$ says that $-T = m\ddot{x}$. Combining this with the given $dm/dt = b\ddot{x}$ gives $mdm = -bTdt$. Integration yields $m^2/2 = C - bTt$. Since $m = M$ when $t = 0$ we have $C = M^2/2$. Therefore,

$$m(t) = \sqrt{M^2 - 2bTt}. \tag{4.152}$$

This holds for $t < M^2/2bT$ (provided that the bucket hasn't hit the wall yet).

(b) The given equation $dm/dt = b\ddot{x} = bdv/dt$ integrates to $v = m/b + C$. Since $v = 0$ when $t = 0$ we have $C = -M/b$. Therefore,

$$v(t) = \frac{m - M}{b} = \frac{\sqrt{M^2 - 2bTt}}{b} - \frac{M}{b}. \tag{4.153}$$

At the instant just before all the sand leaves the bucket, we have $m = 0$. Therefore, $v = -M/b$.
Integrating $v(t)$ to obtain $x(t)$, we find

$$x(t) = \frac{-(M^2 - 2bTt)^{3/2}}{3b^2T} - \frac{M}{b}t + L + \frac{M^3}{3b^2T}, \tag{4.154}$$

where the constant of integration has been chosen to satisfy $x = L$ when $t = 0$.

(c) Using eq. (4.153), the kinetic energy at time t is

$$E = \frac{1}{2}mv^2 = \frac{1}{2b^2}m(m - M)^2. \tag{4.155}$$

Taking the derivative dE/dm to find the maximum, we obtain

$$m = \frac{M}{3} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{2M^3}{27b^2}. \tag{4.156}$$

(d) Using eq. (4.153), the momentum at time t is

$$p = mv = \frac{1}{b}m(m - M). \tag{4.157}$$

Taking the derivative to find the maximum, we obtain

$$m = \frac{M}{2} \qquad \Longrightarrow \qquad p_{\text{max}} = \frac{M^2}{4b}. \tag{4.158}$$

(e) We want $m(t) = \sqrt{M^2 - 2bTt}$ to become zero right when x reaches 0. So we want $x = 0$ when $t = M^2/2bT$. Eq. (4.154) then gives

$$0 = \frac{-M}{b}\left(\frac{M^2}{2bT}\right) + L + \frac{M^3}{3b^2T} \qquad \Longrightarrow \qquad b = \sqrt{\frac{M^3}{6TL}}. \tag{4.159}$$

19. **Right angle in billiards**

Let $\mathbf{v}$ be the initial velocity. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the final velocities. Conservation of momentum and energy give

$$\begin{aligned} m\mathbf{v} &= m\mathbf{v}_1 + m\mathbf{v}_2, \\ \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v}) &= \frac{1}{2}m(\mathbf{v}_1 \cdot \mathbf{v}_1) + \frac{1}{2}m(\mathbf{v}_2 \cdot \mathbf{v}_2). \end{aligned} \tag{4.160}$$

Substituting the $\mathbf{v}$ from the first equation into the second, and using $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1+\mathbf{v}_2)=\mathbf{v}_1\cdot\mathbf{v}_1+2\mathbf{v}_1\cdot\mathbf{v}_2+\mathbf{v}_2\cdot\mathbf{v}_2$ gives

$$\mathbf{v}_1\cdot\mathbf{v}_2=0. \tag{4.161}$$

In other words, the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is 90°. (Or $\mathbf{v}_1=\mathbf{0}$, which means the incoming mass stops because the collision is head-on. Or $\mathbf{v}_2=\mathbf{0}$, which means the masses miss each other.)

20. **Bouncing and recoiling**

Let v_i be the speed of the ball after the ith bounce. Let V_i be the speed of the block right after the ith bounce. Then conservation of momentum gives

$$mv_i = MV_{i+1} - mv_{i+1}. \tag{4.162}$$

Theorem (4.3) says $v_i = V_{i+1} + v_{i+1}$. So we find (the usual result)

$$v_{i+1} = \frac{M-m}{M+m}v_i \equiv \frac{1-\epsilon}{1+\epsilon}v_i \approx (1-2\epsilon)v_i, \tag{4.163}$$

where $\epsilon \equiv m/M \ll 1$. (Likewise, $V_i \approx 2\epsilon v_i$, to leading order in ϵ.) So the speed of the ball after the nth bounce is

$$v_n = (1-2\epsilon)^n v_0 \approx e^{-2n\epsilon}v_0. \tag{4.164}$$

The total distance traveled by the block is easily obtained from energy conservation. Eventually, the ball has negligible energy, so all of its initial kinetic energy goes into heat from friction. Therefore, $mv_0^2/2 = F_f d = (\mu Mg)d$. So

$$d = \frac{mv_0^2}{2\mu Mg}. \tag{4.165}$$

To find the total time, we can add up the times, t_n, after each bounce. Since force times time is the change in momentum, we have $F_f t_n = MV_n$, and so $(\mu Mg)t_n = M(2\epsilon v_n) = 2M\epsilon e^{-2n\epsilon}v_0$. Therefore,

$$\begin{aligned} t=\sum_{n=1}^{\infty} t_n &= \frac{2\epsilon v_0}{\mu g}\sum_{n=1}^{\infty} e^{-2n\epsilon} \\ &= \frac{2\epsilon v_0}{\mu g}\frac{1}{1-e^{-2\epsilon}} \\ &\approx \frac{2\epsilon v_0}{\mu g}\frac{1}{1-(1-2\epsilon)} \\ &= \frac{v_0}{\mu g}. \end{aligned} \tag{4.166}$$

Note that this t is independent of the masses. Also, note that it is much larger than the result obtained in the case where the ball sticks to the block on the first hit (in which case the answer is $mv_0/(\mu Mg)$).

The calculation of d above can also be done by adding up the geometric series of the distances moved after each bounce.

21. **Drag force on a sheet**

(a) We will set $v = 0$ here. If the sheet hits a particle, then the particle acquires a speed essentially equal to $2V$ (by using Theorem (4.3), or by working in the frame of the heavy sheet), and hence a momentum of $2mV$. In time t, the sheet sweeps through a volume AVt, where A is the area of the sheet. Therefore, in time t, the sheet hits $AVtn$ particles. The sheet therefore loses momentum at a rate of $dP/dt = (AVn)(2mV)$. But $F = dP/dt$, so the force per unit area is

$$\frac{F}{A} = 2mnV^2 \equiv 2\rho V^2, \tag{4.167}$$

where ρ is the mass density of the particles. This depends quadratically on V.

(b) For $v \gg V$, particles now hit the sheet on both sides. Note that we can't set V exactly equal to zero here, because we would obtain a result of zero and miss the lowest-order effect. In solving this problem, we need only consider the particles' motions in the x-direction.

The particles in front of the sheet bounce off with speed $v_x + 2V$ (from Theorem (4.3), with the initial relative speed $v_x + V$). So the change in momentum of these particles is $m(2v_x + 2V)$. The rate at which the sheet hits them is $A(v_x + V)(n/2)$, from the reasoning in part (a). ($n/2$ is the relevant density, because half of the particles move to the right, and half move to the left.)

The particles in back of the sheet bounce off with speed $v_x - 2V$ (from Theorem (4.3), with the initial relative speed $v_x - V$). So the change in momentum of these particles is $m(2v_x - 2V)$. The rate at which the sheet hits them is $A(v_x - V)(n/2)$.

Therefore, the force slowing the sheet down is

$$\frac{F}{A} = \frac{1}{A}\frac{dP}{dt} = \Big((n/2)(v_x+V)\Big)\Big(m(2v_x+2V)\Big) - \Big((n/2)(v_x-V)\Big)\Big(m(2v_x-2V)\Big). \tag{4.168}$$

The leading term in this is

$$\frac{F}{A} = 4nmv_xV \equiv 4\rho v_x V, \tag{4.169}$$

where $v_x = v/\sqrt{3}$. This depends linearly on V.

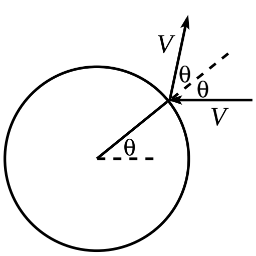


Figure 4.37

22. **Drag force on a cylinder**

Consider a particle which makes contact with the cylinder at an angle θ. In the frame of the (heavy) cylinder (see Fig. 4.37), the particle comes in with speed $-V$ and then bounces off with a horizontal speed $V\cos 2\theta$. So in the lab frame, the particle increases its horizontal momentum by $mV(1 + \cos 2\theta)$.

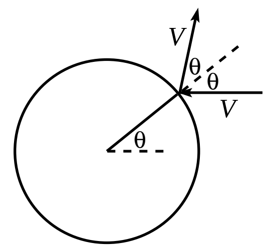
The of area on the cylinder between θ and $\theta + d\theta$ sweeps out a volume at a rate of $(Rd\theta\cos\theta)V\ell$, where ℓ is the length of the cylinder. (The $\cos\theta$ factor gives the projection orthogonal to the direction of motion.)

The force per unit length on the cylinder (that is, the rate of change of momentum, per unit length) is therefore

$$\begin{aligned}\frac{F}{\ell} &= \int_{-\pi/2}^{\pi/2} \Big(n(Rd\theta\,\cos\theta)V\Big)\Big(mV(1+\cos 2\theta)\Big)\\ &= 2nmRV^2\int_{-\pi/2}^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta\end{aligned}$$

$$
\begin{aligned}
&= 2nmRV^2\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{8}{3}nmRV^2 \equiv \frac{8}{3}\rho RV^2. \qquad (4.170)
\end{aligned}
$$

Note that the force per cross-sectional area, $F/(2R\ell)$, equals $(4/3)\rho V^2$. This is less than the result for the sheet in the previous problem, as it should be, since the particles bounce off somewhat sideways from the cylinder.



sphere frame

Figure 4.38

23. **Drag force on a sphere**

Consider a particle which makes contact with the sphere at an angle θ. In the frame of the (heavy) sphere (see Fig. 4.38), the particle comes in with speed $-V$ and then bounces off with a horizontal speed $V\cos 2\theta$. So in the lab frame, the particle increases its horizontal momentum by $mV(1+\cos 2\theta)$.

The of area on the sphere between θ and $\theta + d\theta$ (which is a ring of radius $R\sin\theta$) sweeps out a volume at a rate of $(2\pi R\sin\theta)(Rd\theta)\cos\theta\, V$. (The $\cos\theta$ factor gives the projection orthogonal to the direction of motion.)

The force on the sphere (that is, the rate of change of momentum) is therefore

$$
\begin{aligned}
F &= \int_0^{\pi/2}(n2\pi R^2\sin\theta\cos\theta V)mV(1+\cos 2\theta)d\theta \\
&= 2\pi nmR^2V^2\int_0^{\pi/2}\sin\theta\cos\theta(1+\cos 2\theta)d\theta \\
&= 2\pi nmR^2V^2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}+\frac{\sin 4\theta}{4}\right)d\theta \\
&= \pi nmR^2V^2 \equiv \rho\pi R^2V^2. \qquad (4.171)
\end{aligned}
$$

Note that the force per cross-sectional area, $F/(\pi R^2)$, equals ρV^2. This is less than the results in the two previous problems, as it should be, since the particles bounce off in a more sideways manner from the sphere.

24. **Basketball and tennis ball**

(a) Just before B_1 hits the ground, both balls are moving downward with speed $v = \sqrt{2gh}$ (from $mv^2/2 = mgh$). Just after B_1 hits the ground, it moves upward with speed v, while B_2 is still moving downward with speed v. The relative speed is therefore $2v$. After the balls bounce off each other, the relative speed is still $2v$, from Theorem (4.3). Since the speed of B_1 stays essentially equal to v, the upward speed of B_2 is therefore $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$, or

$$H = d + 9h. \qquad (4.172)$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with speed $v = \sqrt{2gh}$.

We will inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? The relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. The final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$, so

$$v_{i+1} = 2v_i + v. \qquad (4.173)$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{4.174}$$

which is easily seen to satisfy eq. (4.173), with the initial value $v_1 = v$.
From conservation of energy, B_n will bounce to a height of $H = \ell + (2^n - 1)^2 v^2/(2g)$, or

$$H = \ell + (2^n - 1)^2 h. \tag{4.175}$$

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming ℓ is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

REMARK: Escape velocity from the earth is reached when $n = 14$. Of course, the elasticity assumption is absurd in this case, as is the notion that one may find 14 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{14}$. ♣

25. **Colliding masses**

(a) The initial energy of M is $Mv^2/2$. By conservation of momentum, the final speed of the combined masses is $Mv/(M+m) \approx (1 - m/M)v$. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(1 - \frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}mv^2, \\ E_M &= \frac{1}{2}M\left(1 - \frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}Mv^2 - mv^2. \end{aligned} \tag{4.176}$$

The missing energy, $mv^2/2$, is lost to heat.

(b) In this frame, m has initial speed v, so its initial energy is $mv^2/2$. By conservation of momentum, the final speed of the combined masses is $mv/(M+m) \approx (m/M)v$. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right)^2 E \approx 0, \\ E_M &= \frac{1}{2}M\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right) E \approx 0. \end{aligned} \tag{4.177}$$

The missing energy, $mv^2/2$, is lost to heat, in agreement with part (a).

26. **Pulling a chain**

Let x be the distance your hand has moved. Then $x/2$ is the length of the moving part of the chain. The momentum of this part is therefore $p = (\sigma x/2)\dot{x}$. $F = dp/dt$ gives $F = \sigma\dot{x}^2/2 + \sigma x\ddot{x}$. Since v is assumed to be constant, the $\ddot{x}$ term vanishes. (The change in momentum here is due to additional mass acquiring speed v, not due to an increase in speed.) Hence,

$$F = \frac{\sigma v^2}{2}, \tag{4.178}$$

which is constant. Your hand applies this force over a distance $2L$, so the total work you do is

$$F(2L) = \sigma L v^2. \tag{4.179}$$

The total mass of the chain is σL, so the final kinetic energy of the chain is $(\sigma L)v^2/2$. This is only half of the work you did. Therefore, an energy of $(\sigma L)v^2/2$ is lost to heat.

Each atom goes abruptly from rest to speed v, and there is no way to avoid heat loss in such a process. This is quite clear when viewed in the reference frame of your hand. In that frame, the chain initially moves at speed v and eventually comes to rest, piece by piece.

27. **Pulling a rope**

Let x be the position of the end of the rope. Then the momentum of the rope is $(\sigma x)\dot{x}$. Integrating $F = dp/dt$, and using the fact that F is constant, gives $Ft = p = (\sigma x)\dot{x}$. Separating variables and integrating yields

$$\begin{aligned} \int_0^x \sigma x\, dx &= \int_0^t Ft\, dt \\ \Longrightarrow \qquad \frac{\sigma x^2}{2} &= \frac{Ft^2}{2} \\ \Longrightarrow \qquad x &= t\sqrt{F/\sigma}\,. \end{aligned} \tag{4.180}$$

The position therefore grows linearly with time. That is, the speed is constant, and it equals $\sqrt{F/\sigma}$.

REMARK: Realistically, when you grab the rope, there is some small initial value of x (call it ϵ). The dx integral above now starts at ϵ instead of 0, and x takes the form $x = \sqrt{Ft^2/\sigma + \epsilon^2}$. If ϵ is very small, the speed very quickly approaches $\sqrt{F/\sigma}$.

Even if ϵ is not very small, the position becomes arbitrarily close to $t\sqrt{F/\sigma}$, as t becomes large. The "head-start" of ϵ will therefore not help you, in the long run. In retrospect, this has to be the case, since the phrase "if ϵ is not very small" makes no sense, because there is no natural length-scale in the problem which we can compare ϵ with. The only other length that can be formed is $t\sqrt{F/\sigma}$, so any effects of ϵ must arise through a series expansion in the dimensionless quantity $\epsilon/(t\sqrt{F/\sigma})$, which goes to zero for large t, independent of the value of ϵ. ♣

28. **Falling rope**

(a) **First Solution:** Let σ be the density of the rope. By conservation of energy, we may say that the rope's final kinetic energy, $(\sigma L)v^2/2$, equals its loss in potential energy, which is $(\sigma L)(L/2)g$ (because the center-of-mass falls a distance $L/2$). Therefore,

$$v = \sqrt{gL}\,. \tag{4.181}$$

This is the same speed as that obtained by an object that falls a distance $L/2$. If the initial piece hanging down through the hole is arbitrarily short, then the rope will, of course, take an arbitrarily long time to fall down. But the final speed will be always be (arbitrarily close to) $\sqrt{gL}$.

Second Solution: Let σ be the density of the rope. Let x be the length that hangs down through the hole. The $F = ma$ gives $(\sigma x)g = (\sigma L)\ddot{x}$. Therefore, $\ddot{x} = (g/L)x$, and the general solution for x is

$$x = Ae^{t\sqrt{g/L}} + Be^{-t\sqrt{g/L}}. \tag{4.182}$$

(If ϵ is the initial value for x, then $A = B = \epsilon/2$ will satisfy the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$. But we won't need this information in what follows.)

Let t_f be the time for which $x(t_f) = L$. If ϵ is very small (more precisely, if $\epsilon \ll L$), then t_f will be very large. We may therefore neglect the negative-exponent term in eq. (4.182) for this t_f. We then have $L \approx Ae^{t_f\sqrt{g/L}}$. Hence,

$$\dot{x}(t_f) = \sqrt{g/L}\,Ae^{t_f\sqrt{g/L}} = \sqrt{g/L}\,(L) = \sqrt{gL}\,. \tag{4.183}$$

(b) Let σ be the density of the rope. Let x be the length that hangs down through the hole. Then the force on the rope is $(\sigma x)g$. The momentum of the rope is $(\sigma x)\dot{x}$. Therefore, $F = dp/dt$ gives

$$xg = x\ddot{x} + \dot{x}^2. \tag{4.184}$$

($F = ma$ would give the wrong equation, because it neglects the fact that m is changing. It therefore misses the last term in eq. (4.184).)

Since g is the only parameter in eq. (4.184), the solution for $x(t)$ can involve only g's and t's. (The other dimensionful quantities in this problem, L and σ, do not appear in 4.184, so they cannot appear in the solution.) By dimensional analysis, $x(t)$ must therefore be of the form $x(t) = bgt^2$, where b is a numerical constant. Plugging this into eq. (4.184) and dividing by g^2t^2 gives $b = 2b^2 + 4b^2$, and so $b = 1/6$. Our solution may therefore be written as

$$x(t) = \frac{1}{2}\left(\frac{g}{3}\right)t^2. \tag{4.185}$$

This is the equation for something that accelerates downward with acceleration $g' = g/3$. The time the rope takes to fall a distance L is $t = \sqrt{2L/g'} = \sqrt{6L/g}$. The final speed in then

$$v = g't = \sqrt{2Lg'} = \sqrt{2gL/3}\,. \tag{4.186}$$

This is less than the $\sqrt{gL}$ result from part (a). We therefore see that although the total time for the scenario in part (a) is very large, the final speed is still larger than that in the present scenario.

Remarks: From eq. (4.186), we see that 1/3 of the available potential energy is lost to heat. This loss occurs during the abrupt motions that suddenly bring the atoms from zero to non-zero speed when they join the moving part of the rope. Using conservation of energy, therefore, is *not* a valid way to solve this problem.

You can show that the speed in part (a)'s scenario is smaller than the speed in part (b)'s scenario for x less than $2L/3$, but larger for x greater than $2L/3$. ♣

29. **Raising the rope**

Let y be the height of the top of the rope. Let $F(y)$ be the desired force applied by your hand. The net force on the moving part of the rope is $F - (\sigma y)g$, with upward taken to be positive. The momentum of the rope is $(\sigma y)\dot{y}$. Equating the net force to the change in momentum gives

$$\begin{aligned} F - \sigma y g &= \frac{d(\sigma y\dot{y})}{dt} \\ &= \sigma y\ddot{y} + \sigma\dot{y}^2. \end{aligned} \tag{4.187}$$

But $\ddot{y} = 0$, and $\dot{y} = v$. Therefore,

$$F = \sigma y g + \sigma v^2. \tag{4.188}$$

The work that you do is the integral of this force, from $y = 0$ to $y = L$. Hence,

$$W = \frac{\sigma L^2 g}{2} + \sigma L v^2. \tag{4.189}$$

The final potential energy of the rope is $(\sigma L)g(L/2)$, because the center-of-mass is at height $L/2$. This is the first term in eq. (4.189). The final kinetic energy is $(\sigma L)v^2/2$. Therefore, the missing energy $(\sigma L)v^2/2$ is converted into heat. (This is clear, when viewed in the reference frame of your hand. The whole rope is initially moving with speed v, and eventually it all comes to rest.)

30. **The raindrop**

Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

The mass of the raindrop is $M = (4/3)\pi r^3 \rho$. Therefore,

$$\dot{M} = 4\pi r^2 \dot{r} \rho = 3M\frac{\dot{r}}{r}. \tag{4.190}$$

Another expression for $\dot{M}$ is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v \lambda. \tag{4.191}$$

The force of Mg on the droplet equals the rate of change of its momentum, namely $dp/dt = d(Mv)/dt = \dot{M}v + M\dot{v}$. Therefore,

$$Mg = \dot{M}v + M\dot{v}. \tag{4.192}$$

We now have three equations involving the three unknowns, r, M, and v.

(**Note:** We *cannot* write down the naive conservation-of-energy equation, because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.)

The goal is to find $\dot{v}$ for large t. We will do this by first finding $\ddot{r}$ at large t. Eqs. (4.190) and (4.191) give

$$v = \frac{4\rho}{\lambda}\dot{r} \qquad \Longrightarrow \qquad \dot{v} = \frac{4\rho}{\lambda}\ddot{r}. \tag{4.193}$$

Plugging eqs. (4.190) and (4.193) into eq. (4.192) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \tag{4.194}$$

Therefore,

$$\frac{g\lambda}{\rho} r = 12\dot{r}^2 + 4r\ddot{r}. \tag{4.195}$$

Given that the raindrop falls with constant acceleration at large times, we may write [10]

$$\ddot{r} \approx bg, \qquad \dot{r} \approx bgt, \qquad \text{and} \qquad r \approx \frac{1}{2}bgt^2, \tag{4.196}$$

[10] We may justify the constant-acceleration statement in the following way. For large t, let r be proportional to t^α. Then the left side of eq. (4.195) goes like t^α, while the right side goes like $t^{2\alpha-2}$. If these are to be equal, then we must have $\alpha = 2$. Hence, $r \propto t^2$, and $\ddot{r}$ is a constant (for large t).

for large t, where b is a numerical factor to be determined. Plugging eqs. (4.196) into eq. (4.195) gives

$$\left(\frac{g\lambda}{\rho}\right)\left(\frac{1}{2}bgt^2\right) = 12(bgt)^2 + 4\left(\frac{1}{2}bgt^2\right)bg. \tag{4.197}$$

Therefore, $b = \lambda/28\rho$. Hence, $\ddot{r} = g\lambda/28\rho$, and eq. (4.193) gives the acceleration of the raindrop at large t,

$$\dot{v} = \frac{g}{7}, \tag{4.198}$$

independent of ρ and λ.

REMARK: We can calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen.

The fact that v is proportional to $\dot{r}$ (shown in eq. (4.193)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is 1/4 of the way from the base to the apex. Therefore, if M is the mass of the raindrop after it has fallen a height h, then the loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{4.199}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{4.200}$$

where ΔE_{int} is the gain in internal thermal energy. The energy required to heat 1g of water by 1 degree is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by 1 degree is ≈ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{4.201}$$

where mks units are used, and T is measured in celsius. (We have assumed that the internal energy is uniformly distributed throughout the raindrop.) Eqs. (4.200) and (4.201) give the increase in temperature as a function of h,

$$4200\, \Delta T = \frac{3}{28}gh. \tag{4.202}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100\,^\circ\text{C}$ is found to be

$$h = 400\,\text{km}. \tag{4.203}$$

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h is 10 km, which would raise the temperature by two or three degrees. This effect, of course, is washed out by many other factors. ♣