Chapter 3

Oscillations

In this chapter we will discuss oscillatory motion. The simplest examples of such motion are a swinging pendulum and a mass attached to the end of a spring, but it is possible to make the system more complicated by introducing a damping force and/or an external driving force. We will study all of these cases.

We are interested in oscillatory motion for two reasons. First, we study it because we *can* study it. This is one of the few systems in physics where we can solve for the motion completely. (There's nothing wrong with looking under the lamppost every now and then.) Second, such systems are ubiquitous in physics, for reasons that will become clear in Section 4.2. If there was ever a type of physical system worthy of study, this is it.

We'll jump right into some math at the beginning of this chapter. Then we'll show how the math is applied to the physics.

3.1 Linear differential equations

A *linear differential equation* is one in which x and its time derivatives enter only through their first powers. An example is $3d^4x/dt^4 + 7dx/dt + x = 0$. If the right-hand side of the equation is zero, then we use the term *homogeneous* differential equation. If the right-hand side is some function of t (as in the case of $3\ddot{x} - 4\dot{x} = 9t^2 - 5$), then we use the term *inhomogeneous* differential equation. The goal of this chapter is to learn how to solve these types of equations. Linear differential equations come up again and again in physics, so we had better find a systematic method of solving them.

The techniques that we will need are best learned through examples, so let's solve a few differential equations, starting with some simple ones. Throughout this chapter, x will be understood to be a function of t. Hence, a dot will denote time differentiation.

Example 1 ($\dot{x} = ax$): This is a very simple differential equation. There are (at least) two ways to solve it.

First method: Separate variables to obtain $dx/x = adt$, and then integrate to obtain $\ln x = at + c$. Exponentiate to obtain

$$x = Ae^{at}, \tag{3.1}$$

where $A \equiv e^c$ is a constant factor. A is determined by the value of x at, say, $t = 0$.

Second method: A better method is to just look at $\dot{x} = ax$ and realize that $x = Ae^{at}$ solves it. This may seem a bit silly. And in addition to being cheap, it is also rather restrictive. But as we will see below, guessing these exponential functions (or sums of them) is actually the most general thing we can try, so the method is indeed quite general.

REMARK: Using this method, you may be concerned that although you have found one solution to the equation, you might have missed another one. But the general theory of differential equations says that a first-order linear equation has only one independent solution (we'll just accept this fact here). So if you find one solution, you know that you've found the whole thing. ♣

Example 2 ($\ddot{x} = ax$): If a is negative, then this equation describes the oscillatory motion of, say, a spring (about which we'll have much more to say later). If a is positive, then it describes exponentially growing or decaying motion. There are (at least) three ways to solve this equation.

First method: You can use the method of Section 2.3 here, because our system is one where the force depends on only the position x. But this method is rather cumbersome. It will certainly work, but in the case where our equation is a *linear* function of x, there are much simpler methods.

Second method: A better method is to look at the equation and just write down the answer. (Again, this may seem cheap. But the fact of the matter is that most of the equations you'll see look basically the same, so you might as well keep using this cheap method if it keeps working.) The cases $a > 0$ and $a < 0$ are slightly different, so let's write the equation as

$$\ddot{x} = \pm\omega^2 x, \tag{3.2}$$

where ω is a real number (which we assume to be positive.)

In the case of the minus sign, we simply note that any multiple of $\cos\omega t$, $\sin\omega t$, $e^{i\omega t}$, or $e^{-i\omega t}$ solves the equation. So the general solution may be written in various equivalent forms:

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t, \\ x(t) &= C\cos(\omega t + \phi_1), \\ x(t) &= D\sin(\omega t + \phi_2), \\ x(t) &= Ee^{i\omega t} + Fe^{-i\omega t}. \end{aligned} \tag{3.3}$$

The various constants are related to each other; for example, $A = C\cos\phi_1$, and $B = -C\sin\phi_1$. Note that there are two free parameters in each of the above expressions for $x(t)$. These parameters are determined by the initial conditions (say, the position and speed at $t = 0$). Depending on the specifics of a given problem, one of the above forms will work better than the others.

In the case of the plus sign in eq. (3.2), we simply note that any multiple of $\cosh \omega t$, $\sinh \omega t$, $e^{\omega t}$, or $e^{-\omega t}$ solves the equation. So the general solution may be written in various equivalent forms:

$$\begin{aligned} x(t) &= A\cosh \omega t + B \sinh \omega t, \\ x(t) &= C\cosh(\omega t + \phi_1), \\ x(t) &= D\sinh(\omega t + \phi_2), \\ x(t) &= Ee^{\omega t} + Fe^{-\omega t}. \end{aligned} \tag{3.4}$$

(If you're unfamiliar with the hyperbolic trig functions, a few facts are listed in Appendix A.)

Again, the general theory of differential equations says that our second-order linear equation has only two independent solutions. Therefore, having found two solutions, we know we've found them all.

There was actually no need to separate the $\pm\omega^2$ cases. Written as $\ddot{x} = ax$, the general solution is $x(t) = A\cosh\sqrt{a}\,t + B\sinh\sqrt{a}\,t$. If a happens to be negative, then $\sqrt{a}$ is imaginary (call it $i\alpha$). Hence, the hyperbolic trig functions turn into ordinary trig functions, because $\cosh i\alpha = \cos\alpha$, and $\sinh i\alpha = i\sin\alpha$.

VERY IMPORTANT REMARK: The fact that the sum of two different solutions is again a solution to our equation (a fact that we used in writing, for example, eq. (3.3)) is a monumentally important property of *linear* differential equations. This property does *not* hold for nonlinear differential equations, e.g. $\ddot{x}^2 = x$, because in this case squaring after adding two solutions produces a cross-term which destroys the equality.

This property is called the *principle of superposition*. That is, superimposing two solutions yields another solution. This quality makes theories in physics that are governed by linear equations *much* easier to deal with than ones that are governed by nonlinear ones. General Relativity, for example, is permeated by nonlinear equations, and solutions to most General Relativity systems are extremely difficult to come by.

For equations with one main condition
(Those linear), we give you permission
To take your solutions,
With firm resolutions,
And add them in superposition. ♣

Third method: The third method is the most general, and it is the one we will use repeatedly in this chapter. The procedure (for a homogeneous equation) is to guess a solution of the form $x(t) = Ae^{\alpha t}$, and then find out what α must be. (You can't solve for A because it cancels out of the equation, since the equation is linear in x and homogeneous. A is determined by the initial conditions.) Plugging $Ae^{\alpha t}$ into $\ddot{x} = ax$ gives $\alpha = \pm\sqrt{a}$. We have therefore found two solutions. The most general solution is an arbitrary linear combination of these,

$$x(t) = Ae^{\sqrt{a}\,t} + Be^{-\sqrt{a}\,t}. \tag{3.5}$$

A and B are determined by the initial conditions.

Of course, you could just look at $\ddot{x} = ax$ and write down the solution in eq. (3.5) (this is simply the second method above). But the method of trying a solution of the form $x(t) = Ae^{\alpha t}$ will be needed in cases where the solution is not so obvious.

REMARK: If a is negative (in which case let's define $a \equiv -\omega^2$, where ω is a real number), then the solution looks like $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$, this can be written in terms of trig functions, if desired (see eq. (3.3)).

If a is positive (in which case let's define $a \equiv \omega^2$, where ω is a real number), then the solution looks like $x(t) = Ae^{\omega t} + Be^{-\omega t}$. Using $e^{\theta} = \cosh\theta + \sinh\theta$, this can be written in terms of hyperbolic trig functions, if desired (see eq. (3.4)).

Although the solution in eq. (3.5) is completely correct for both signs of a, it is generally more illuminating to write the negative-a solutions in either the trig form or the $e^{\pm i\omega t}$ exponential form where the i's are explicit. ♣

The usefulness of this third method cannot be overemphasized. It may seem somewhat restrictive, but it works. The examples in the remainder of this chapter should convince you of this.

This is our method, essential,
For equations we solve, differential.
It gets the job done,
And it's even quite fun.
We just try a routine exponential.

Example 3 ($\ddot{x} + 2\gamma\dot{x} + ax = 0$): This will be our last mathematical example, then we'll get into some physics. As we will see later, this example pertains to a damped harmonic oscillator. We have put a factor of 2 in the coefficient of $\dot{x}$ in order to make some later formulas look nicer.

Note that the force in this example, which is $-2\gamma\dot{x} - ax$ (times m), depends on both v and x, so our methods of Section 2.3 don't apply. This leaves us with either the method of clever guessing or the method of trying $Ae^{\alpha t}$. We're probably not going to guess this one, so let's apply our lovely method trying $Ae^{\alpha t}$.

Plugging $Ae^{\alpha t}$ into the given equation, and canceling the nonzero factor of $Ae^{\alpha t}$, yields

$$\alpha^2 + 2\gamma\alpha + a = 0. \tag{3.6}$$

The solutions for α are

$$-\gamma \pm \sqrt{\gamma^2 - a}. \tag{3.7}$$

Call these α_1 and α_2. Then the general solution to our equation is

$$\begin{aligned} x(t) &= Ae^{\alpha_1 t} + Be^{\alpha_2 t} \\ &= e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2 - a}} + Be^{-t\sqrt{\gamma^2 - a}}\right). \end{aligned} \tag{3.8}$$

(Hmmm, our method of trying $Ae^{\alpha t}$ doesn't look so trivial anymore . . .)

If $\gamma^2 - a < 0$, then we can write our answer in terms of sines and cosines, and we have oscillatory motion which decreases in time due to the $e^{-\gamma t}$ factor (or it increases, if $\gamma < 0$, but this is rarely physical). If $\gamma^2 - a > 0$, then we have exponential motion.

In general, if we have a linear differential equation of the type

$$\frac{d^n x}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + c_1\frac{dx}{dt} + c_0 x = 0, \tag{3.9}$$

then the strategy is to simply plug in $x(t) = Ae^{\alpha t}$ and (in theory) solve the resulting nth order equation (namely $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0$), for α, to obtain the solutions $\alpha_1, \ldots, \alpha_n$. The general solution for $x(t)$ is then

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \cdots + A_n e^{\alpha_n t}, \tag{3.10}$$

where the A_i are determined by the initial conditions. In practice, however, we will rarely encounter differential equations of degree higher than 2. (Note: if some of the α_i happen to be equal, then (3.10) is not valid. We will encounter such a situation in Section 3.2.2.)

3.2 Oscillatory motion

3.2.1 Simple harmonic motion

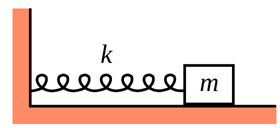

Figure 3.1

Let's now do some real live physical problems. We'll start with simple harmonic motion. This is the motion undergone by a particle subject to a force $F(x) = -kx$.

The classic system that undergoes simple harmonic motion is a mass attached to a spring (see Fig. 3.1). A typical spring has a force of the form $F(x) = -kx$, where x is the displacement from equilibrium. (This holds as long as the spring isn't stretched too far; eventually this expression breaks down for any real spring.)

Hence, $F = ma$ gives $-kx = m\ddot{x}$, or

$$\ddot{x} + \omega^2 x = 0, \qquad \text{where} \;\; \omega \equiv \sqrt{\frac{k}{m}}\,. \tag{3.11}$$

From Example 2 in the previous section, the solution to this may be written as in eq. (3.3),

$$x(t) = C\cos(\omega t + \phi). \tag{3.12}$$

The system therefore oscillates back and forth forever in time.

REMARK: The constants C and ϕ are determined by the initial conditions. If, for example, $x(0) = 0$ and $\dot{x}(0) = v$, then we must have $0 = C\cos\phi$ and $v = -C\omega\sin\phi$. Hence, $\phi = \pi/2$, and $C = -v/\omega$. Therefore, the solution is $x(t) = -(v/\omega)\cos(\omega t + \pi/2)$. This looks a little nicer as $x(t) = (v/\omega)\sin(\omega t)$. ♣

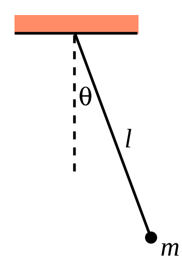

Figure 3.2

Example (Simple pendulum): Another classic system that undergoes (approximate) simple harmonic motion is the simple pendulum, that is, a mass that hangs on a massless string and swings in a vertical plane.

Let ℓ be the length of the string. Let θ be the angle the string makes with the vertical (see Fig. 3.2). Then the gravitational force on the mass in the tangential direction is $-mg\sin\theta$. So $F = ma$ in the tangential direction gives

$$-mg\sin\theta = m(\ell\ddot{\theta}) \tag{3.13}$$

(The tension in the string exactly cancels the radial component of gravity, so the radial $F = ma$ gives us no relevant information.) We will now enter the realm of approximations and assume that the amplitude of the oscillations is small. This allows us to write $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \omega^2\theta = 0, \qquad \text{where} \quad \omega \equiv \sqrt{\frac{g}{\ell}}. \tag{3.14}$$

Therefore,

$$\theta(t) = C\cos(\omega t + \phi). \tag{3.15}$$

The true motion is arbitrarily close to this, for sufficiently small amplitudes. Exercise 1 deals with the higher-order corrections to the motion in the case where the amplitude is not tiny.

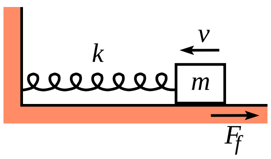

Figure 3.3

3.2.2 Damped harmonic motion

Consider a mass m attached to the end of a spring which has a spring constant k. Let the mass be subject to a drag force proportional to its velocity, $F_f = -bv$ (see Fig. 3.3). What is the position as a function of time?

The force on the mass is $F = -b\dot{x} - kx$. So $F = m\ddot{x}$ gives

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \tag{3.16}$$

where $2\gamma \equiv b/m$, and $\omega \equiv \sqrt{k/m}$. But this is exactly the equation we solved in Example 3 in the previous section. Now, however, we have the restrictions $\gamma > 0$, and $\omega^2 > 0$. Letting $\Omega^2 \equiv \gamma^2 - \omega^2$, for simplicity, we may write the solution in eq. (3.8) as

$$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right), \qquad \text{where} \quad \Omega^2 \equiv \gamma^2 - \omega^2. \tag{3.17}$$

There are three cases to consider.

Case 1: Underdamping ($\Omega^2 < 0$)

In this case, $\omega > \gamma$. Since Ω is imaginary, let us define $\Omega \equiv i\tilde{\omega}$ (so $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$) We then have

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right) \\ &\equiv e^{-\gamma t}C\cos(\tilde{\omega}t + \phi). \end{aligned} \tag{3.18}$$

These two forms are equivalent. Depending on the circumstances of the problem, one form works better than the other. (Or perhaps one of the other forms in eq. (3.3) will be the most useful one, to be multiplied by the $e^{-\gamma t}$ factor.) The constants are related by $A + B = C\cos\phi$ and $A - B = iC\sin\phi$. In a physical problem, $x(t)$ is real, so we must have $A^* = B$ (where the star denotes complex conjugation). The two constants A and B, or C and ϕ, are determined by the initial conditions.

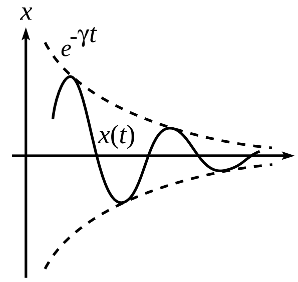

Figure 3.4

The cosine form makes it apparent that the motion is harmonic motion whose amplitude decreases in time because of the $e^{-\gamma t}$ factor. A plot of such motion is shown in Fig. 3.4. Note that the frequency of the motion, $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$, is less than the natural frequency, ω, of the undamped oscillator.

REMARK: If γ is very small, then $\tilde{\omega} \approx \omega$, which makes sense, because we almost have an undamped oscillator. If γ is very close to ω, then $\tilde{\omega} \approx 0$. So the oscillations are very slow. Of course, for very small $\tilde{\omega}$ it's hard to even tell that the oscillations exist, since they will damp out on a time scale of order $1/\gamma$, which will be short compared to the time scale of the oscillations, $1/\tilde{\omega}$. ♣

Case 2: Overdamping ($\Omega^2 > 0$)

In this case, $\omega < \gamma$. Ω is real (and taken to be positive), so we have

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}. \tag{3.19}$$

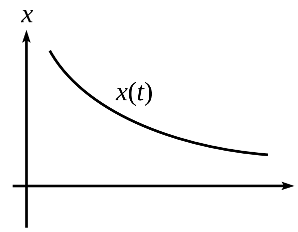

Figure 3.5

There is no oscillatory motion in this case (see Fig. 3.5). Note that $\gamma > \Omega \equiv \sqrt{\gamma^2 - \omega^2}$, so both of the exponents are negative. The motion therefore goes to zero for large t. (This had better be the case. A real spring is not going to have the motion go off to infinity. If we had obtained a positive exponent somehow, we'd know we had made a mistake.)

REMARK: If γ is just slightly larger than ω, then $\Omega \approx 0$, so the two terms in (3.19) are roughly equal, and we essentially have exponential decay, according to $e^{-\gamma t}$. If $\gamma \gg \omega$ (that is, strong damping), then $\Omega \approx \gamma$, so the first term in (3.19) dominates, and we essentially have exponential decay according to $e^{-(\gamma-\Omega)t}$. We can be somewhat quantitative about this by approximating Ω by $\Omega \equiv \sqrt{\gamma^2 - \omega^2} = \gamma\sqrt{1 - \omega^2/\gamma^2} \approx \gamma(1 - \omega^2/2\gamma^2)$. Therefore, the exponential behavior goes like $e^{-\omega^2 t/2\gamma}$. This is slow decay (that it, slow compared to $t \sim 1/\omega$), which makes sense if the damping is very strong. ♣

Case 3: Critical damping ($\Omega^2 = 0$)

In this case, $\gamma = \omega$. Eq. (3.16) therefore becomes $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$. In this special case, we have to be careful in solving our differential equation. The solution in eq. (3.17) is not valid, because in the procedure leading to eq. (3.8), the roots α_1 and α_2 are equal (to $-\gamma$). (So we have really found only one solution, $e^{-\gamma t}$.) We'll just invoke here the result from the theory of differential equations which says that in this special case, the other solution is of the form $te^{-\gamma t}$.

REMARK: You should check explicitly that $te^{-\gamma t}$ solves the equation $\ddot{x}+2\gamma\dot{x}+\gamma^2 x = 0$. Or if you want to, you can derive it in the spirit of Problem 1. In the more general case where there are n identical roots in the procedure leading to eq. (3.10) (call them all α), the n independent solutions to the differential equation are $t^k e^{\alpha t}$, for $0 \le k \le (n-1)$. But more often than not, there are no repeated roots, so you don't have to worry about all this. ♣

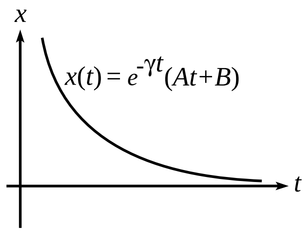

Figure 3.6

Our solution is therefore of the form

$$x(t) = e^{-\gamma t}(A + Bt). \tag{3.20}$$

The exponential factor eventually wins out over the Bt term, of course, so the motion goes to zero for large t (see Fig. 3.6).

If we are given a spring with a fixed ω, and if we look at the system at different values of γ, then critical damping (when $\gamma = \omega$) is the case where the motion converges to zero in the quickest way (which is like $e^{-\omega t}$). This is true because in the underdamped case ($\gamma < \omega$), the envelope of the oscillatory motion goes like $e^{-\gamma t}$, which goes to zero slower than $e^{-\omega t}$, since $\gamma < \omega$. And in the overdamped case ($\gamma > \omega$), the dominant piece is the $e^{-(\gamma-\Omega)t}$ term. And as you can verify, if $\gamma > \omega$ then $\gamma - \Omega \equiv \gamma - \sqrt{\gamma^2 - \omega^2} < \omega$, so this motion also goes to zero slower than $e^{-\omega t}$.

3.2.3 Driven (and damped) harmonic motion

Mathematical prelude

Before we examine driven harmonic motion, we have to learn how to solve a new type of differential equation. How can we solve something of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = C_0 e^{i\omega_0 t}, \tag{3.21}$$

where ω_0 is a given quantity? This is an inhomogeneous differential equation, due to the term on the right-hand side. It's not very physical, since the right-hand side is complex, but we're doing math now. Equations of this sort will come up again and again, and fortunately there is a nice, easy (although sometimes messy) method for solving them. As usual, the method is to make a reasonable guess, plug it in, and see what condition comes out.

Since we have the $e^{i\omega_0 t}$ sitting on the right side, let's try a solution of the form $x(t) = Ae^{i\omega_0 t}$. (A will depend on ω_0, among other things, as we will see.) Plugging this into eq. (3.21), and canceling the non-zero factor of $e^{i\omega_0 t}$, we obtain

$$(-\omega_0^2)A + 2\gamma(i\omega_0)A + aA = C_0. \tag{3.22}$$

Solving for A, we find our solution for x to be

$$x(t) = \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}. \tag{3.23}$$

Note the differences between this technique and the one in Example 3 in Section 3.1. In that example, the goal was to determine what the α in $x(t) = Ae^{\alpha t}$ had to be. And there was no way to solve for A; the initial conditions determined A. But in the present technique, the ω_0 in $x(t) = Ae^{i\omega_0 t}$ is a given quantity, and the goal is to solve for A in terms of the given constants. Therefore, in the solution in eq. (3.23), there are *no free constants* to be determined by initial conditions. We've found one particular solution, and we're stuck with it. (The term *particular solution* is what people use for eq. (3.23).)

With no freedom to adjust the solution in eq. (3.23), how can we satisfy an arbitrary set of initial conditions? Fortunately, eq. (3.23) does not represent the most general solution to eq. (3.21). The most general solution is the sum of our particular solution in eq. (3.23), *plus* the 'homogeneous' solution we found in eq. (3.8). This is obvious, because the solution in eq. (3.8) was explicitly constructed to yield zero when plugged into the left-hand side of eq. (3.21). Therefore, tacking it onto our particular solution won't change the equality in eq. (3.21), because the left side is linear. The principle of superposition has saved the day.

The complete solution to eq. (3.21) is therefore

$$x(t) = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right) + \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right)e^{i\omega_0 t}, \tag{3.24}$$

where A and B are determined by the initial conditions.

It is clear what the strategy should be if we have a slightly more general equation to solve, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}. \tag{3.25}$$

Simply solve the equation with only the first term on the right. Then solve the equation with only the second term on the right. Then add the two solutions. And then add on the homogeneous solution from eq. (3.8). We are able to apply the principle of superposition because the left-hand side of our equation is linear.

Finally, let's look at the case where we have many such terms on the right-hand side, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}. \tag{3.26}$$

We simply have to solve N different equations, each with just one of the N terms on the right-hand side. Then add up all the solutions, then add on the homogeneous solution from eq. (3.8). If N is infinite, that's fine. You'll just have to add up an infinite number of solutions. This is the principle of superposition at its best.

REMARK: The previous paragraph (which is only applicable because the left-hand side of (3.26) is linear), combined with a basic result from Fourier analysis, allows us to solve (in principle) any equation of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = f(t). \tag{3.27}$$

Fourier analysis says that any (nice enough) function, $f(t)$, may be decomposed into its Fourier components,

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}. \tag{3.28}$$

In this continuous sum, the functions $g(\omega)$ take the place of the coefficients C_n in eq. (3.26). So, if $S_\omega(t)$ is the solution for $x(t)$ when there is only the term $e^{i\omega t}$ on the right-hand side of eq. (3.27) (that is, $S_\omega(t)$ is the solution given in eq. 3.23)), then the complete particular solution to (3.27) is

$$x(t) = \int_{-\infty}^{\infty} g(\omega) S_\omega(t). \tag{3.29}$$

Finding the coefficients $g(\omega)$ is the hard part (or, rather, the messy part), but we won't bother getting into that here. We won't do anything with Fourier analysis in this course, but we just wanted to let you know that it *is* possible solve (3.27) for any function $f(t)$. Most of the functions we'll consider will be nice functions like $\cos\omega_0 t$, for which the Fourier decomposition is simply $\cos\omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$. ♣Let's now do a physical example.

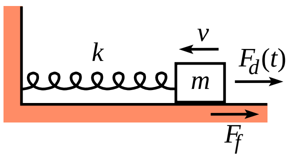

Figure 3.7

Example (Driven spring): Consider a spring with spring constant k. A mass m at the end of the spring is subject to a friction force proportional to its velocity, $F_f = -bv$. The mass is also subject to a driving force, $F_d(t) = F_d \cos\omega_d t$ (see Fig. 3.7). What is the position as a function of time?

Solution: The force on the mass is $F(x, \dot{x}, t) = -b\dot{x} - kx + F_d \cos\omega_d t$. So $F = ma$ gives

$$\begin{aligned} \ddot{x} + 2\gamma\dot{x} + \omega^2 x &= F\cos\omega_d t \\ &= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right). \end{aligned} \tag{3.30}$$

where $2\gamma \equiv b/m$, $\omega \equiv \sqrt{k/m}$, and $F \equiv F_d/m$. Using eq. (3.23) and the technique of adding solutions mentioned after eq. (3.25), our particular solution is

$$x_p(t) = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right) e^{i\omega_d t} + \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right) e^{-i\omega_d t}. \tag{3.31}$$

The complete solution is the sum of this particular solution and the homogeneous solution from eq. (3.17).

Let's simplify eq. (3.31) a bit. Getting the i's out of the denominators, and turning the exponentials into sines and cosines, we find (as you can show)

$$x_p(t) = \left(\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\sin\omega_d t. \tag{3.32}$$

Note that this is real, as it must be, if it is to describe the position of a particle.

REMARK: If you wish, you can solve eq. (3.30) by simply taking the real part of the solution to eq. (3.21) (that is, the $x(t)$ in eq. (3.23)), because $\mathrm{Re}(e^{i\omega_d t}) = \cos(\omega_d t)$. It is clear that (with $C_0 = F$) the real part of eq. (3.23) does indeed give eq. (3.32), because in eq. (3.31) we've just taken half of a quantity plus its complex conjugate, which is the real part.

If you don't like using complex numbers, another way of solving eq. (3.30) is to keep it in the form with the $\cos\omega_d t$ on the right, and then simply guess a solution of the form $A\cos\omega_d t + B\cos\omega_d t$, and solve for A and B. The result will be eq. (3.32). ♣

We can simplify eq. (3.32) a bit further. If we define

$$R \equiv \sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}, \tag{3.33}$$

then we may rewrite eq. (3.32) as

$$\begin{aligned} x_p(t) &= \frac{F}{R}\left(\frac{(\omega^2-\omega_d^2)}{R}\cos\omega_d t + \frac{2\gamma\omega_d}{R}\sin\omega_d t\right) \\ &= \frac{F}{R}\cos(\omega_d t - \phi), \end{aligned} \tag{3.34}$$

where ϕ is defined by

$$\cos\phi = \frac{\omega^2-\omega_d^2}{R}\,, \sin\phi = \frac{2\gamma\omega_d}{R} \qquad \Longrightarrow \qquad \tan\phi = \frac{2\gamma\omega_d}{\omega^2-\omega_d^2}\,. \tag{3.35}$$

(Note that $0 < \phi < \pi$, since $\sin\phi$ is positive.)

Recalling the solution in eq. (3.17), we may write the complete solution to eq. (3.30) as

$$x(t) = \frac{F}{R}\cos(\omega_d t - \phi) + e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right). \tag{3.36}$$

The constants A and B are determined by the initial conditions. Note that if there is any damping at all in the system (that is, $\gamma > 0$), then the homogeneous part of the solution goes to zero for large t, and we are left with only the particular solution. In other words, the system approaches a definite $x(t)$, independent of the initial conditions.

REMARK: The amplitude of the solution in eq. (3.34) is proportional to $1/R = [(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2]^{-1/2}$. Given ω_d and γ, this is maximum when $\omega = \omega_d$. Given ω and γ, this is maximum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$; in the case of weak damping (that is, $\gamma \ll \omega$), the maximum is achieved when $\omega_d \approx \omega$. The term *resonance* is used to describe this situation where the natural frequency of the driving force is picked just right to make the amplitude of the oscillations as large as possible. Note that the phase angle ϕ equals $\pi/2$ when $\omega_d \approx \omega$; the motion of the particle lags the driving force by a quarter of a cycle. ♣

3.3 Coupled oscillators

Mathematical prelude

In the previous sections, we have dealt with only one function of time, $x(t)$. What if we have two functions of time, say $x(t)$ and $y(t)$, which are related by a pair of "coupled" differential equations? For example,

$$\begin{aligned} 2\ddot{x} + \omega^2(5x - 3y) &= 0, \\ 2\ddot{y} + \omega^2(5y - 3x) &= 0. \end{aligned} \tag{3.37}$$

We'll assume $\omega^2 > 0$ here, but this isn't necessary. We call these equations "coupled" because there are x's and y's in both of them, and it is not immediately obvious how to separate them to solve for x and y. There are (at least) two methods of solving these equations.

First method: Sometimes it is easy, as in this case, to find certain linear combinations of the given equations for which nice things happen. Taking the sum, we find

$$(\ddot{x}+\ddot{y})+\omega^2(x+y)=0. \tag{3.38}$$

This equation involves x and y only in the combination of their sum, $x+y$. The solution is

$$x+y=A_1\cos(\omega t+\phi_1), \tag{3.39}$$

where A_1 and ϕ_1 are determined by initial conditions. (It could also be written as a sum of exponentials, or the sum of a sine and cosine, of course.) We may also take the difference, to find

$$(\ddot{x}-\ddot{y})+4\omega^2(x-y)=0. \tag{3.40}$$

This equation involves x and y only in the combination of their difference, $x-y$. The solution is

$$x-y=A_2\cos(2\omega t+\phi_2), \tag{3.41}$$

Taking the sum and difference of eqs. (3.39) and (3.41), we find

$$\begin{aligned} x(t) &= B_1\cos(\omega t+\phi_1)+B_2\cos(2\omega t+\phi_2), \\ y(t) &= B_1\cos(\omega t+\phi_1)-B_2\cos(2\omega t+\phi_2), \end{aligned} \tag{3.42}$$

where the B_i are half of the A_i.

The strategy of this solution was simply to fiddle around and try to form differential equations that involve the same combination of the variables on both sides, such as eqs. (3.38) and (3.40). Then you just call this combination by the new name, "z", if you wish, and write down the obvious solution for z, as in eqs. (3.39) and (3.41).

We've managed to solve our equations for x and y. However, the more interesting thing we've done is produce the equations (3.39) and (3.41). The combinations $(x+y)$ and $(x-y)$ are called the *normal coordinates* of the system. These are the combinations that oscillate with one pure frequency. The motion of x and y will, in general, look rather complicated, and it may be difficult to tell that the motion is really made up of just the two frequencies in eq. (3.42). But if you plot the values of $(x+y)$ and $(x-y)$ as time goes by, then you will find nice sinusoidal graphs, even if x and y are each behaving in a rather unpleasant manner.

Second method: In the event that it is not easy to guess what linear combinations of eqs. (3.37) will yield equations involving just one combination of x and y (the $x+y$ and $x-y$ above), there is a fail-proof method for solving for x and y. In the spirit of Section 3.1, let us try a solution of the form $x=Ae^{i\alpha t}$ and $y=Be^{i\alpha t}$, which we will write (for convenience) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}. \tag{3.43}$$

It is not obvious that there should exist solutions for x and y which have the same t dependence, but let's try it and see what happens. Note that we've explicitly put the i in the exponent, but there's no loss of generality here. If α happens to be imaginary, then the exponent is real. It's personal preference whether or not you put the i in.

Plugging our guess into eqs. (3.37), and dividing through by $e^{i\omega t}$, we find

$$\begin{aligned} 2A(-\alpha^2) + 5A\omega^2 - 3B\omega^2 &= 0, \\ 2B(-\alpha^2) + 5B\omega^2 - 3A\omega^2 &= 0, \end{aligned} \tag{3.44}$$

or equivalently, in matrix form,

$$\begin{pmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.45}$$

This homogeneous equation for A and B has a nontrivial solution (that is, one where A and B aren't both 0) only if the matrix is *not* invertible (because if it were invertible, we could just multiply through by the inverse to obtain $(A, B) = (0, 0)$).

When is a matrix invertible? There is a straightforward (although tedious) method for finding the inverse of a matrix. It involves taking cofactors, taking a transpose, and dividing by the determinant. The step that concerns us here is the division by the determinant. The inverse will exist if and only if the determinant is not zero. So we see that eq. (3.45) has a nontrivial solution only if the determinant is zero. Since we seek a nontrivial solution, we must demand that

$$\begin{aligned} 0 &= \begin{vmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{vmatrix} \\ &= 4\alpha^4 - 20\alpha^2\omega^2 + 16\omega^4. \end{aligned} \tag{3.46}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm 2\omega$. We have therefore found four types of solutions. If $\alpha = \pm\omega$, then we can plug this back into eq. (3.45) to obtain $A = B$. (Both equations give this same result. This was essentially the point of setting the determinant equal to 0.) If $\alpha = \pm 2\omega$, then eq. (3.45) gives $A = -B$. (Again, the equations are redundant.) Note that we cannot solve exactly for A and B, but only for their ratio. Adding up our four solutions, we see that x and y take the general form (written in vector form for the sake of simplicity and bookkeeping),

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &+ A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2i\omega t}. \end{aligned} \tag{3.47}$$

The four A_i are determined from the initial conditions.

We can rewrite eq. (3.47) in a somewhat cleaner form. If the coordinates x and y describe the positions of particles, they must be real. Therefore, A_1 and A_2 must be complex conjugates, and likewise for A_3 and A_4. If we then define

$A_2^* = A_1 \equiv (B_1/2)e^{i\phi_1}$ and $A_4^* = A_3 \equiv (B_2/2)e^{i\phi_2}$, we may rewrite the solution in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\omega t + \phi_2), \tag{3.48}$$

where the B_i and ϕ_i are real (and are determined from the initial conditions). We have therefore reproduced the result in eq. (3.42).

It is clear from eq. (3.48) that the combinations $x + y$ and $x - y$ (the normal coordinates) oscillate with pure frequencies ω and 2ω, respectively.

It is also clear that if $B_2 = 0$, then $x = y$ at all times, and they both oscillate with frequency ω. And if $B_1 = 0$, then $x = -y$ at all times, and they both oscillate with frequency 2ω. These two pure-frequency motions are called the *normal modes*. They are labeled by the vectors $(1, 1)$ and $(1, -1)$. The significance of normal modes will become clear in the following example.

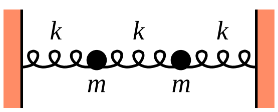

Figure 3.8

Example (Two masses, three springs): Consider two masses, m, connected to each other and to two walls by three springs, as shown in Fig. 3.8. The three springs have the same spring constant k. Find the positions of the masses as functions of time. What are the normal coordinates? What are the normal modes?

Solution: Let $x_1(t)$ and $x_2(t)$ be the positions of the left and right masses, respectively, relative to their equilibrium positions. The middle spring is stretched a distance $x_2 - x_1$. Therefore, the force on the left mass is $-kx_1 + k(x_2 - x_1)$, and the force on the right mass is $-kx_2 - k(x_2 - x_1)$. (It's easy to make a mistake on the sign of the second term in these expressions. You can double check the sign by, say, looking at the force when x_2 is very big.) Therefore, $F = ma$ on each mass gives (with $\omega^2 = k/m$)

$$\begin{aligned} \ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ \ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0. \end{aligned} \tag{3.49}$$

These are rather friendly coupled equations, and we can see that the sum and difference are the useful combinations to take. The sum gives

$$(\ddot{x}_1 + \ddot{x}_2) + \omega^2(x_1 + x_2) = 0, \tag{3.50}$$

and the difference gives

$$(\ddot{x}_1 - \ddot{x}_2) + 3\omega^2(x_1 - x_2) = 0. \tag{3.51}$$

The solutions to these equations are the normal coordinates,

$$\begin{aligned} x_1 + x_2 &= A_+ \cos(\omega t + \phi_+), \\ x_1 - x_2 &= A_- \cos(\sqrt{3}\omega t + \phi_-). \end{aligned} \tag{3.52}$$

Taking the sum and difference of these normal coordinates, we have

$$\begin{aligned} x_1(t) &= B_+ \cos(\omega t + \phi_+) + B_- \cos(\sqrt{3}\omega t + \phi_-), \\ x_2(t) &= B_+ \cos(\omega t + \phi_+) - B_- \cos(\sqrt{3}\omega t + \phi_-), \end{aligned} \tag{3.53}$$

where the B's are half the A's.

REMARK: We may also derive eqs. (3.53) by using the determinant method. Letting $x_1 = Ae^{i\alpha t}$ and $x_2 = Be^{i\alpha t}$, we see that for there to be a nontrivial solution for A and B, we must have

$$\begin{aligned} 0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= \alpha^4 - 4\alpha^2\omega^2 + 3\omega^4. \end{aligned} \tag{3.54}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm\sqrt{3}\omega$. If $\alpha = \pm\omega$, then eq. (3.49) yields $A = B$. If $\alpha = \pm\sqrt{3}\omega$, then eq. (3.49) yields $A = -B$. The solutions for x_1 and x_2 therefore take the general form

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &\quad + A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\sqrt{3}i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\sqrt{3}i\omega t} \\ &\to B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_+) + B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_-). \end{aligned} \tag{3.55}$$

This is equivalent to eq. (3.53). ♣

The normal modes are obtained by setting either B_- or B_+ equal to zero in eq. (3.53). Therefore, they are $(1,1)$ and $(1,-1)$. How do we visualize these? The mode $(1,1)$ oscillates with frequency ω. In this case (where $B_- = 0$), we have $x_1(t) = x_2(t)$, at all times. So the masses simply oscillate back and forth in the same manner, as shown in Fig. 3.9. It is clear that such motion has frequency ω, because as far as the masses are concerned, the middle spring is not there, so each mass moves under the influence of just one spring, and hence with frequency ω.

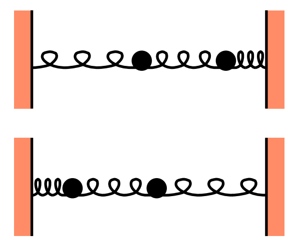

Figure 3.9

The mode $(1,-1)$ oscillates with frequency $\sqrt{3}\omega$. In this case (where $B_+ = 0$), we have $x_1(t) = -x_2(t)$, at all times. So the masses oscillate back and forth with opposite displacements, as shown in Fig. 3.10. It is clear that this mode should have a frequency larger than that of the other mode, because the middle spring is being stretched, so the masses feel a larger force. But it takes a little thought to show that the frequency is $\sqrt{3}\omega$.

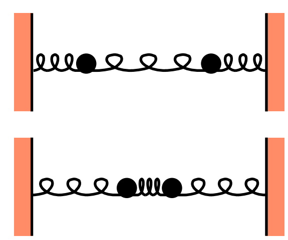

Figure 3.10

REMARK: The normal mode $(1,1)$ above is associated with the normal coordinate $x_1 + x_2$; they both involve the frequency ω. However, this association is *not* due to the fact that the coefficients of both x_1 and x_2 in this normal coordinate are equal to 1. Rather, it is due to the fact that the *other* normal mode (namely $(x_1, x_2) \propto (1,-1)$) gives no contribution to the sum $x_1 + x_2$.

There are a few too many 1's floating around in the above example, so it's hard to see what results are meaningful and what results are coincidence. The following example should clear things up. Let's say we solved a problem using the determinant method, and we found the solution to be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \cos(\omega_2 t + \phi_2). \tag{3.56}$$

Then $5x + y$ is the normal coordinate associated with the normal mode $(3, 2)$, which has frequency ω_1. And $2x - 3y$ is the normal coordinate associated with the normal mode $(1, -5)$, which has frequency ω_2. ♣

REMARK: Note the difference between the types of differential equations we solved in Section 2.3 of the previous chapter, and the types we solved in this chapter. The former dealt with forces that did not have to be linear in x or $\dot{x}$, but which had to depend on only x, or only $\dot{x}$, or only t. The latter dealt with forces that could depend on all three of these quantities, but which had to be linear in x and $\dot{x}$. ♣

3.4 Exercises

Section 3.2: Oscillatory motion

1. **Corrections to the pendulum** ∗∗∗

   (a) For small oscillations, the period of a pendulum is approximately $T \approx 2\pi\sqrt{\ell/g}$, independent of amplitude, θ_0. For finite oscillations, show that the exact expression for T is

   $$T = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}\,. \tag{3.57}$$

   (b) Let's now find an approximation to this value of T. It's more convenient to deal with quantities that go to 0 as $\theta \to 0$, so make use of the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$ to write T in terms of sines. Then make the change of variables, $\sin x \equiv \sin(\theta/2)/\sin(\theta_0/2)$. Finally, expand your integrand judiciously in powers of (the fairly small quantity) θ_0, and perform the integrals to show

   $$T \approx 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right). \tag{3.58}$$

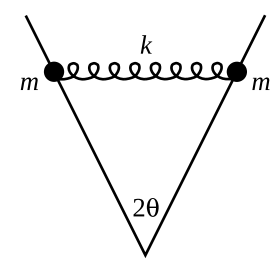

Figure 3.11

2. **Angled rails**

   Two particles of mass m are constrained to move along two rails which make an angle of 2θ with respect to each other, as shown in Fig. 3.11. They are connected by a spring with spring constant k. What is the frequency of oscillations for the motion where the spring remains parallel to its equilibrium position?

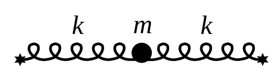

Figure 3.12

3. **Springs all over** ∗∗

   (a) A mass m is attached to two springs which have equilibrium lengths equal to zero. The other ends of the springs are fixed at two points (see Fig. 3.12). The spring constants are the same. The mass rests at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Ignore gravity.)

   (b) A mass m is attached to a number of springs which have equilibrium lengths equal to zero. The other ends of the springs are fixed at various points in space (see Fig. 3.13). The spring constants are all the same. The mass rests at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Ignore gravity.)

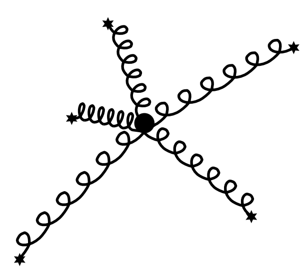

Figure 3.13

Section 3.3: Coupled oscillators

4. **Springs between walls** ∗∗

   Four identical springs and three identical masses lie between two walls (see Fig. 3.14). Find the normal modes.

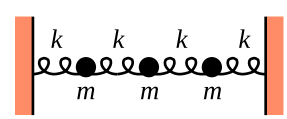

Figure 3.14

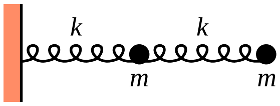

Figure 3.15

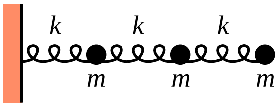

Figure 3.16

5. **Springs and one wall** $**$

   (a) Two identical springs and two identical masses are attached to a wall as shown in Fig. 3.15. Find the normal modes.

   (b) Three identical springs and three identical masses are attached to a wall as shown in Fig. 3.16. Find the normal modes.

3.5 Problems

Section 3.1: Linear differential equations

1. **A limiting case** $*$

   Consider the equation $\ddot{x} = ax$. If $a = 0$, then the solution to $\ddot{x} = 0$ is of course $x(t) = C + Dt$. Show that in the limit $a \to 0$, eq. (3.5) reduces to this form. *Note:* $a \to 0$ is a very sloppy way of saying what we mean. What is the precise mathematical condition we should write?

Section 3.2: Oscillatory motion

2. **Exponential force**

   A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are 0. Find $x(t)$.

   (This problem was already given in Chapter 2, but solve it here in the spirit of Section 3.2.3.)

3. **Average tension** $**$

   Is the average (over time) tension in the string of a pendulum larger or smaller than mg? How much so? (As usual, assume that the angular amplitude, A, is small.)

4. **Through the circle** $**$

   A very large plane (consider it to be infinite), of mass density σ (per area), has a hole of radius R cut in it. A particle initially sits in the center of the circle, and is then given a tiny kick perpendicular to the plane. Assume that the only force acting on the particle is the gravitational force from the plane. Find the frequency of small oscillations (that is, where the amplitude is small compared to R).

Section 3.3: Coupled oscillators

5. **Springs on a circle** $****$

   (a) Two identical masses are constrained to move on a circle. Two identical springs connect the masses and wrap around a circle (see Fig. 3.17). Find the normal modes.

   (b) Three identical masses are constrained to move on a circle. Three identical springs connect the masses and wrap around a circle (see Fig. 3.18). Find the normal modes.

   (c) How about the general case with N identical masses and N identical springs?

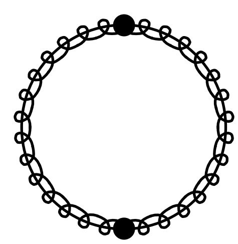

Figure 3.17

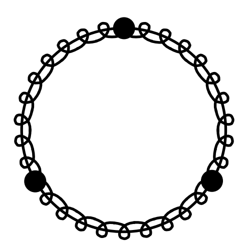

Figure 3.18

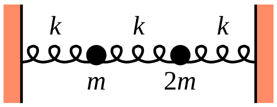

Figure 3.19

6. **Unequal masses** $**$

Three identical springs and two masses, m and $2m$, lie between two walls as shown in Fig. 3.19. Find the normal modes.

3.6 Solutions

1. **A limiting case**

   The statement $a \to 0$ is nonsensical, because a has units of $[\text{time}]^{-2}$, and the number 0 has no units. The proper statement is that eq. (3.5) reduces to $x(t) = C + Dt$ when t satisfies $t \ll 1/\sqrt{a}$. Both sides of this relation have units of time. Under this condition, $\sqrt{a}\,t \ll 1$, so we may write $e^{\pm\sqrt{a}\,t}$ approximately as $1 \pm \sqrt{a}\,t$. Therefore, eq. (3.5) becomes

$$\begin{aligned} x(t) &\approx A(1 + \sqrt{a}\,t) + B(1 - \sqrt{a}\,t) \\ &= (A + B) + \sqrt{a}(A - B)t \\ &\equiv C + Dt \end{aligned} \tag{3.59}$$

   If C and D happen to be of order 1 in the units chosen, then A and B must be roughly negatives of each other, and both of order $1/\sqrt{a}$.

   If a is small but nonzero, then t will eventually become large enough so that the linear form in eq. (3.59) is not valid.

2. **Exponential force**

   Guess a particular solution to $\ddot{x} = e^{-bt}$ of the form $x(t) = Ce^{-bt}$. Then $C = 1/b^2$. The solution to the homogeneous equation $\ddot{x} = 0$ is $x(t) = At + B$. Therefore, the complete solution for x is $x(t) = e^{-bt}/b^2 + At + B$. The initial conditions are $0 = v(0) = -1/b + A$, and $0 = x(0) = 1/b^2 + B$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}. \tag{3.60}$$

3. **Average tension**

   Let the length of the pendulum be ℓ. We know that the angle, θ, depends on time according to

$$\theta(t) = A\cos(\omega t), \tag{3.61}$$

   where $\omega = \sqrt{g/\ell}$, and A is small. The tension, T, in the string must account for the radial component of gravity, $mg\cos\theta$, plus the centripetal acceleration, $m\ell\dot{\theta}^2$. Using eq. (3.61), this gives

$$T = mg\cos\Big(A\cos(\omega t)\Big) + m\ell\Big(-\omega A\sin(\omega t)\Big)^2. \tag{3.62}$$

   Using the small-angle approximation $\cos\alpha \approx 1 - \alpha^2/2$, we have (since A is small)

$$\begin{aligned} T &\approx mg\left(1 - \frac{1}{2}A^2\cos^2(\omega t)\right) + m\ell\omega^2A^2\sin^2(\omega t) \\ &= mg + mgA^2\left(\sin^2(\omega t) - \frac{1}{2}\cos^2(\omega t)\right). \end{aligned} \tag{3.63}$$

   The average value of $\sin^2\theta$ and $\cos^2\theta$ over one period is $1/2$, so the average value for T is

$$\overline{T} = mg + \frac{1}{2}mgA^2, \tag{3.64}$$

   which is larger than mg, by $mgA^2/2$.

   Note that it is quite reasonable to expect $\overline{T} > mg$, because the average value of the vertical component of T equals mg (since the pendulum has no net rise or fall over a long period of time), and there is some positive contribution from the horizontal component of T.

4. **Through the circle**

By symmetry, only the component of the gravitational force perpendicular to the plane will survive. Let the particle's coordinate relative to the plane be z, and let its mass be m. Then a piece of mass dm at radius r on the plane will provide a force equal to $Gm(dm)/(r^2+z^2)$. To get the component perpendicular to the plane, we must multiply this by $z/\sqrt{r^2+z^2}$. So the total force on the particle is

$$\begin{aligned} F(z) &= -\int_R^\infty \frac{\sigma G m z 2\pi r\, dr}{(r^2+z^2)^{3/2}} \\ &= 2\pi\sigma G m z (r^2+z^2)^{-1/2}\Big|_{r=R}^{r=\infty} \\ &= -\frac{2\pi\sigma G m z}{\sqrt{R^2+z^2}} \\ &\approx -\frac{2\pi\sigma G m z}{R}, \end{aligned} \tag{3.65}$$

where we have used $z \ll R$. Therefore, $F=ma$ gives

$$\ddot{z} + \frac{2\pi\sigma G}{R} z = 0. \tag{3.66}$$

The frequency of small oscillations is then

$$\omega = \sqrt{\frac{2\pi\sigma G}{R}}. \tag{3.67}$$

Remark: For everyday values of R, this is a rather small number, because G is so small. Let's roughly determine its size. If the sheet has thickness d, and it is made out of a material with density ρ (per volume), then $\sigma = \rho d$. Hence, $\omega = \sqrt{2\pi\rho d G/R}$.

In the above analysis, we assumed the sheet was infinitely thin. In practice, we would need d to be much smaller than the amplitude of the motion. But this amplitude needs to be much smaller than R, in order for our approximations to hold. So we conclude that $d \ll R$. To get a rough upper bound on ω, let's pick $d/R = 1/10$; and let's make ρ be five times the density of water (i.e., $5000\,\text{kg/m}^3$). Then $\omega \approx 5\cdot 10^{-4}\,\text{s}^{-1}$, which corresponds to a little more than one oscillation every 4 hours.

For an analogous system consisting of electrical charges, the frequency is much larger, since the electrical force is so much stronger than the gravitational force. ♣

5. **Springs on a circle**

(a) Pick two equilibrium positions (any diametrically opposite points will do). Let the distances of the masses from these points be x_1 and x_2 (measured counterclockwise). Then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -2k(x_1-x_2), \\ m\ddot{x}_2 &= -2k(x_2-x_1). \end{aligned} \tag{3.68}$$

The determinant method works here, but let's just do it the easy way. Adding these equations gives

$$\ddot{x}_1 + \ddot{x}_2 = 0. \tag{3.69}$$

Subtracting them equations gives

$$(\ddot{x}_1 - \ddot{x}_2) + 4\omega^2(x_1-x_2) = 0. \tag{3.70}$$

The normal coordinates are therefore

$$\begin{aligned} x_1 + x_2 &= At + B, \qquad \text{and} \\ x_1 - x_2 &= C\cos(2\omega t + \phi). \end{aligned} \tag{3.71}$$

And the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi). \end{aligned} \tag{3.72}$$

The first mode has frequency 0, and corresponds to the masses sliding around the circle, equally spaced, at constant speed.

(b) Pick three equilibrium positions (any three equally spaced points will do). Let the distances of the masses from these points be x_1, x_2, and x_3 (measured counterclockwise). Then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - x_2) - k(x_1 - x_3), \\ m\ddot{x}_2 &= -k(x_2 - x_3) - k(x_2 - x_1), \\ m\ddot{x}_3 &= -k(x_3 - x_1) - k(x_3 - x_2). \end{aligned} \tag{3.73}$$

It's easy to see that the sum of these equations gives something nice, Also, differences between any two of the equations gives something useful. But let's use the determinant method to get some practice. Trying solutions proportional to $e^{i\alpha t}$ yields the determinant equation

$$\begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{vmatrix} = 0. \tag{3.74}$$

One solution is $\alpha^2 = 0$. The other solution is the double root $\alpha^2 = 3\omega^2$.
The $\alpha = 0$ root corresponds to the vector $(1, 1, 1)$. So this normal mode is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(At + B). \tag{3.75}$$

This mode has frequency 0, and corresponds to the masses sliding around the circle, equally spaced, at constant speed.
The $\alpha^2 = 3\omega^2$ root corresponds to a two-dimensional subspace of normal modes. You can show that any vector of the form (a, b, c) with $a + b + c = 0$ is a normal mode with frequency $\sqrt{3}\omega$. We will arbitrarily pick the vectors $(0, 1, -1)$ and $(1, 0, -1)$ as basis vectors in this space. We can then write the normal modes as linear combinations of the vectors

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\cos(\sqrt{3}\omega t + \phi_1), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\cos(\sqrt{3}\omega t + \phi_2). \end{aligned} \tag{3.76}$$

REMARKS: This is very similar to the example in section 3.3 with three springs and two masses oscillating between two walls. The way we've written these modes, the first one has the first mass stationary (so there could be a wall there, for all the other two masses know), and the second one has the second mass stationary.

The normal coordinates in this problem are $x_1 + x_2 + x_3$ (obtained by adding the three equations in (3.73)), $x_2 - x_3$ (obtained by subtracting the third eq. in (3.73) from the second), and $x_1 - x_3$ (obtained by subtracting the third eq. in (3.73) from the first). Actually, any combination of the form $ax_1 + bx_2 + cx_3$, with $a + b + c = 0$, is a normal mode (obtained by taking a times the first eq. in (3.73), etc.) ♣

(c) In part (b), what we were essentially doing, by setting the determinant in eq. (3.74) equal to 0, was finding the eigenvectors[1] of the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3I - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.77}$$

We haven't bothered writing the common factor ω^2, since this won't affect the eigenvectors. We'll let the reader show that for the general case of N springs and masses, the above matrix becomes the $N \times N$ matrix

$$3I - \begin{pmatrix} 1 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & 0 \\ 0 & 0 & 1 & 1 & & 0 \\ & \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \equiv 3I - M. \tag{3.78}$$

In M, the three consecutive 1's keep shifting to the right, and they wrap around cyclicly.

Let's now be a little tricky. We can guess the eigenvectors and eigenvalues of M if we take a hint from its cyclic nature. A particular set of things that are rather cyclic are the Nth roots of 1. If η is an Nth root of 1, we leave it to you to show that $(1, \eta, \eta^2, \ldots, \eta^{N-1})$ is a eigenvector of M with eigenvalue $\eta^{-1} + 1 + \eta$. (This general method works for any matrix where the entries keep shifting to the right, and the entries don't even have to be equal.) The eigenvalues of the entire matrix in eq. (3.78) are therefore $3 - (\eta^{-1} + 1 + \eta) = 2 - \eta^{-1} - \eta$.

There are N different Nth roots of 1, namely $\eta_n = e^{2\pi i n/N}$. So the N eigenvalues are

$$\lambda_n = 2 - \left(e^{-2\pi i n/N} + e^{2\pi i n/N}\right) = 2 - 2\cos(2\pi n/N). \tag{3.79}$$

The corresponding eigenvectors are

$$V_n = \left(1, \eta_n, \eta_n^2, \ldots, \eta_n^{N-1}\right). \tag{3.80}$$

The eigenvalues come in pairs. The numbers n and $N - n$ give the same value. This is fortunate, since we may then form real linear combinations of the two

[1] An eigenvector, v, of a matrix, M, is a vector that gets taken into a multiple of itself when acted upon by M. That is, $Mv = \lambda v$, where λ is some number. You can prove for yourself that such a λ must satisfy det $|M - \lambda I| = 0$, where I is the identity matrix. We don't assume a knowledge of eigenvectors in this course, so don't worry about this problem.

corresponding eigenvectors. The vectors

$$V_n^+ \equiv \frac{1}{2}(V_n + V_{N-n}) = \begin{pmatrix} 1 \\ \cos(2\pi n/N) \\ \cos(4\pi n/N) \\ \vdots \\ \cos(2(N-1)\pi n/N) \end{pmatrix} \tag{3.81}$$

and

$$V_n^- \equiv \frac{1}{2i}(V_n - V_{N-n}) = \begin{pmatrix} 0 \\ \sin(2\pi n/N) \\ \sin(4\pi n/N) \\ \vdots \\ \sin(2(N-1)\pi n/N) \end{pmatrix} \tag{3.82}$$

both have eigenvalue λ_n. The frequencies corresponding to these normal modes are

$$\omega_n = \sqrt{\lambda_n} = \sqrt{2 - 2\cos(2\pi n/N)}. \tag{3.83}$$

The only values for which the n's don't pair up are 0, and $N/2$ (if N is even). Let's check our results for $N = 3$. If $n = 0$, we find $\lambda_0 = 0$, and $V_0 = (1, 1, 1)$. If $n = 1$, we find $\lambda_1 = 3$, and $V_1^+ = (1, -1/2, -1/2)$ and $V_1^- = (0, 1/2, -1/2)$. These two vectors span the same space we found in part (b).

6. **Unequal masses**

Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. The equations of motion are

$$\begin{aligned} \ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ 2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0. \end{aligned} \tag{3.84}$$

The appropriate linear combinations of these equations are not obvious, so we'll use the determinant method. Letting $x_1 = Ae^{i\alpha t}$ and $x_2 = Be^{i\alpha t}$, we see that for there to be a nontrivial solution for A and B, we must have

$$\begin{aligned} 0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= 2\alpha^4 - 6\alpha^2\omega^2 + 3\omega^4. \end{aligned} \tag{3.85}$$

The roots of this equation are

$$\alpha = \pm\omega\sqrt{\frac{3+\sqrt{3}}{2}} \equiv \pm\alpha_1, \qquad \text{and} \qquad \alpha = \pm\omega\sqrt{\frac{3-\sqrt{3}}{2}} \equiv \pm\alpha_2. \tag{3.86}$$

If $\alpha^2 = \alpha_1^2$, then the normal mode is proportional to $(\sqrt{3}+1, -1)$. If $\alpha^2 = \alpha_2^2$, then the normal mode is proportional to $(\sqrt{3}-1, 1)$. So the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \phi_1), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \phi_2), \end{aligned} \tag{3.87}$$

Note that these two vectors are not orthogonal. There is no need for them to be.) You can easily show that the normal coordinates are $x_1 - (\sqrt{3}-1)x_2$, and $x_1 + (\sqrt{3}+1)x_2$, respectively.