Chapter 2

Using $F = ma$

2.1 Newton's Laws

The general goal of classical mechanics is to determine what happens to a given set of objects in a given physical situation. In order to figure such things out, we need to know what makes objects move the way they do. This subject goes by the name *dynamics*, and Newton's laws are the starting point. These laws may be stated as follows (there are other possible variations).

- **First Law:** A body moves with constant velocity (which may be zero) unless acted on by a force.
- **Second Law:** The time rate of change of the momentum of a body equals the force acting on the body.
- **Third Law:** The forces two bodies apply to each other are equal in magnitude and opposite in direction.

We could discuss for days on end the degree to which these statements are physical laws, and the degree to which they are definitions. Sir Arthur Eddington once made the unflattering comment that the first law essentially says that "every particle continues in its state of rest or uniform motion in a straight line except insofar that it doesn't." Although Newton's laws may seem somewhat vacuous at first glance, there is actually a bit more content to them than Eddington's statement implies. Let's look at each in turn. The discussion will be brief, because we have to save time for other things in this book that we really *do* want to discuss for days on end.

First Law

One thing this law does is give a definition of zero force.

Another thing it does is give a definition of an *inertial frame* (which is defined simply as a reference frame in which the first law holds). The term 'velocity' is used, so we have to state what frame of reference we are measuring the velocity with respect to. The first law does *not* hold in an arbitrary frame. For example, it fails in the frame of a spinning turntable.[1] Intuitively, an inertial frame is one that moves at constant speed. But this is ambiguous, because you have to say what the frame is moving at constant speed *with respect to.* At any rate, an inertial frame may be defined as the special type of frame where the first law holds.

So, what we have now are two intertwined definitions of 'force' and 'inertial frame'. Not much physical content there. *But*, however sparse in content the law is, it still holds for *all* particles. So if we have a frame where one free particle moves with constant velocity, then if we replace it with another particle, it will likewise move with constant velocity. This is a statement with content.

Second Law

One thing this law does is give a definition of non-zero force. Momentum is defined[2] to be $m\mathbf{v}$. If m is constant, then the law says $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law holds only in an inertial frame (which was defined by the first law).

So far, this law merely gives a definition of $\mathbf{F}$. But the meaningful statement arises when we invoke the fact that the law holds for *all* particles. If the same force (for example, the same spring stretched by the same amount) acts on two particles, with masses m_1 and m_2, then their accelerations are related by

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}. \tag{2.1}$$

This relation holds regardless of what the common force is. Therefore, once you've used one force to find the relative masses of two objects, then you know what the ratio of their a's will be when they are subjected to any other force.

Of course, we haven't really defined *mass* yet. But eq. (2.1) gives an experimental method for determining an object's mass in terms of a standard (say, 1 kg) mass. All you have to do is compare its acceleration with that of the standard mass.

There is also another piece of substance in this law, in that it says $\mathbf{F} = m\mathbf{a}$, instead of, say, $\mathbf{F} = m\mathbf{v}$ or $\mathbf{F} = m\, d^3\mathbf{x}/dt^3$. This issue is related to the first law. $\mathbf{F} = m\mathbf{v}$ is certainly not viable, because the first law says that it is possible to have a velocity without a force. And $\mathbf{F} = md^3\mathbf{x}/dt^3$ would make the first law incorrect, because it would then be true that a particle moves with constant acceleration (instead with constant speed) unless acted on by a force.

Note that $\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really three equations in one. In cartesian coordinates, it says that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

Third Law

This law essentially postulates that momentum is conserved. There isn't much left to be defined in this law, so this statement is one of pure content. It says that if you

[1] Well, it's possible to fudge things so that Newton's laws hold in such a frame, but we'll save the discussion of this for Chapter 9.

[2] We're doing everything nonrelativistically here, of course. Chapter 11 gives the relativistic modification of the $m\mathbf{v}$ expression.

have two isolated particles, then their accelerations are opposite in direction and inversely proportional to their masses.

This third law cannot be a definition, because it's actually not always valid. It only holds for forces of the 'pushing' and 'pulling' type. It fails for the magnetic force, for example.

2.2 Free-body diagrams

The law that allows us to be quantitative is the second law. Given a force, we can apply $\mathbf{F} = m\mathbf{a}$ to find the acceleration. And knowing the acceleration, we should be able to determine the behavior of a given object (that is, where it is and how fast it is moving). This process sometimes takes a bit of work, but there are two basic types of situations that commonly arise.

- In many problems, all you are given is a physical situation (for example, a block resting on a plane, strings connecting masses, etc.), and it is up to you to find all the forces acting on all the objects. These forces generally point in various directions, so it is easy to lose track of them. It therefore proves useful to isolate the objects and draw all the forces acting on each of them. This is the subject of the present section.

- In other problems, you are *given* the force, $F(x)$, as a function of position (we'll just work in one dimension here), and the task immediately becomes the mathematical one of solving the $F(x) = ma \equiv m\ddot{x}$ equation. These *differential equations* can be difficult (or impossible) to solve exactly. They are the subject of Section 2.3.

Let's now consider the first of these two types of problems, where we are presented with a physical situation, and where we must determine all the forces involved. The term *free-body diagram* is used to denote a diagram with all the forces drawn on all the objects. After drawing such a diagram, you simply write down all the $F = ma$ equations it implies. The result will be a system of linear equations in various unknown forces and accelerations. This procedure is best understood through an example.

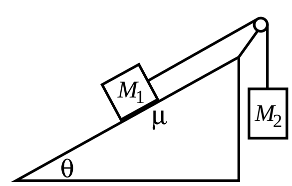

Figure 2.1

Example (A plane and masses): Mass M_1 is held on a plane with inclination angle θ, and mass M_2 hangs over the side. They are connected by a massless string which runs over a massless pulley (see Fig. 2.1). The coefficient of friction (assume the kinetic and static coefficients are equal) between M_1 and the plane is μ. Mass M_1 is released. Assume that M_2 is sufficiently large so that M_1 gets pulled up the plane. What is the acceleration of the system? What is the tension in the string?

Solution: The first thing to do is draw all the forces on the two masses. These are shown in Fig. 2.2. The forces on M_2 are gravity and the tension. The forces on M_1 are gravity, friction, the tension, and the normal force. The friction force points down the plane, since we are assuming that M_1 moves up the plane.

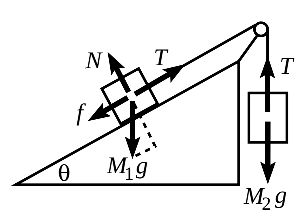

Figure 2.2

We now simply have to write down all the $F = ma$ equations. When dealing with M_1, we could break things up into horizontal and vertical components, but it is much cleaner to use the components tangential and normal to the plane. These two components of $\mathbf{F} = m\mathbf{a}$, along with the vertical $F = ma$ for M_2, give

$$\begin{aligned} T - f - M_1 g \sin\theta &= M_1 a, \\ N - M_1 g\cos\theta &= 0, \\ M_2 g - T &= M_2 a, \end{aligned} \tag{2.2}$$

where we have used the fact that the two masses accelerate at the same rate (and we have defined the positive direction for M_2 to be downward). Also, the tension is the same at both ends of the string, because otherwise there would be a net force on some part of the string which would then have to undergo infinite acceleration, since it is massless.

There are four unknowns: T, a, N, and f. Fortunately, we have a fourth equation, namely $f = \mu N$. Therefore, the second equation above gives $f = \mu M_1 g\cos\theta$. The first equation then becomes $T - \mu M_1 g\cos\theta - M_1 g\sin\theta = M_1 a$. This may be combined with the third equation to give

$$a = \frac{g(M_2 - \mu M_1\cos\theta - M_1\sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 + \mu\cos\theta + \sin\theta). \tag{2.3}$$

Note that we must have $M_2 > M_1(\mu\cos\theta + \sin\theta)$ in order for M_1 to move upward. This is clear from looking at the forces tangential to the plane.

Remark: If we had instead assumed that M_1 was sufficiently large so that it slides down the plane, then the friction force would point up the plane, and we would have found

$$a = \frac{g(M_2 + \mu M_1\cos\theta - M_1\sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 - \mu\cos\theta + \sin\theta). \tag{2.4}$$

In order for M_1 to move downward (i.e., $a < 0$), we must have $M_2 < M_1(\sin\theta - \mu\cos\theta)$. Therefore, $M_1(\sin\theta - \mu\cos\theta) < M_2 < M_1(\mu\cos\theta + \sin\theta)$ is the range of M_2 for which the system doesn't move. ♣

In problems like the one above, it is clear what things you should pick as the objects on which you're going to draw forces. But in other problems, there are various different subsystems you can choose, and you must be careful to include all the relevant forces on a given subsystem. Which subsystems you want to pick depends on what quantities you're trying to find. Consider the following example.

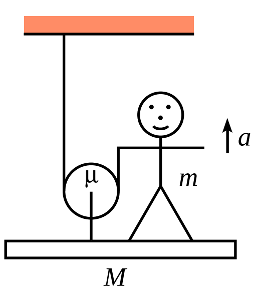

Figure 2.3

Example (Platform and pulley): A person stands on a platform-and-pulley system, as shown in Fig. 2.3. The masses of the platform, person, and pulley are M, m, and μ, respectively.[3] The rope is massless. Let the person pull up on the rope so that she has acceleration a upwards.

(a) What is the tension in the rope?

[3]Assume that the pulley's mass is concentrated at its center, so we don't have to worry about any rotational dynamics (the subject of Chapter 7).

(b) What is the normal force between the person and the platform? What is the tension in the rod connecting the pulley to the platform?

Solution:

(a) To find the tension in the rope, we simply want to let our subsystem be the whole system. If we imagine putting the system in a black box (to emphasize the fact that we don't care about any internal forces within the system), then the forces we see "protruding" from the box are the three weights (Mg, mg, and μg) downward, and the tension T upward. Applying $F = ma$ to the whole system gives

$$T-(M+m+\mu)g=(M+m+\mu)a \qquad \Longrightarrow \qquad T=(M+m+\mu)(g+a). \quad (2.5)$$

(b) To find the normal force, N, between the person and the platform, and also the tension, f, in the rod connecting the pulley to the platform, it is not sufficient to consider the system as a whole. We must consider subsystems.

Let's apply $F = ma$ to the person. The forces acting on the person are gravity, the normal force from the platform, and the tension from the rope (pulling downward on the person at her hand). Therefore, we have

$$N - T - mg = ma. \quad (2.6)$$

Now apply $F = ma$ to the platform. The forces acting on the platform are gravity, the normal force from the person, and the force upwards from the rod. Therefore, we have

$$f - N - Mg = Ma. \quad (2.7)$$

Now apply $F = ma$ to the pulley. The forces acting on the pulley are gravity, the force downward from the rod, and *twice* the tension in the rope (since it pulls up on both sides). Therefore, we have

$$2T - f - \mu g = \mu a. \quad (2.8)$$

Note that if we add up the three previous equations, we obtain the $F = ma$ equation in eq. (2.5), as should be the case, since the whole system is the sum of the three previous subsystems. Eqs. (2.6) – (2.8) are three equations in the three unknowns (T, N, and f). Their sum quickly yields the T in (2.5), and then eqs. (2.6) and (2.8) give, respectively,

$$N = (M + 2m + \mu)(g + a), \qquad \text{and} \qquad f = (2M + 2m + \mu)(g + a), \quad (2.9)$$

as you can show.

Of course, you can also obtain these results by considering subsystems different from the ones we chose above (for example, you might choose the pulley-plus-platform, etc.). But no matter how you choose to break up the system, you will need to produce three independent $F = ma$ statements in order to solve for the three unknowns (T, N, and f).

In problems like this one, it is easy to make a mistake by forgetting to include one of the forces, such as the second T in eq. (2.8). The safest thing to do, therefore, is to isolate each subsystem, draw a box around it, and then write down all the forces that "protrude" from the box. Fig. 2.4 shows the free-body diagram for the subsystem of the pulley.

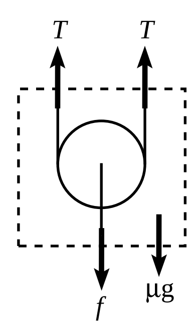

Figure 2.4

2.3 Solving differential equations

Let's now consider the type of problem where we are *given* the force $F(x)$ as a function of position, and where our task is to solve the $F(x) = ma \equiv m\ddot{x}$ differential equation, to find the position as a function of time, $x(t)$. In the present section, we will develop a few techniques for solving differential equations. The ability to apply these techniques dramatically increases the number of problems we can solve.

In general, the force F is a function of the position x, the speed $\dot{x}$, and the time t. (Of course, it could be a function of d^2x/dt^2, d^3x/dt^3, etc., but these cases don't arise much, so we won't worry about them.) We therefore want to solve the differential equation,

$$m\ddot{x} = F(x, \dot{x}, t). \tag{2.10}$$

In general, this equation cannot be solved exactly for $x(t)$.[4] But for most of the problems we will deal with, it can be solved. The problems we'll encounter will often fall into one of three special cases, namely, where F is a function of t only, or x only, or $v \equiv \dot{x}$ only. In all of these cases, one must invoke the given initial conditions, $x_0 \equiv x(t_0)$ and $v_0 \equiv v(t_0)$, which appear in the limits of the integrals in the following discussion.

You may just want to skim the following page and a half, and then refer back to it, as needed. Don't try to memorize all the different steps. We present them only for completeness. The whole point here can basically be summarized by saying that sometimes you want to write $\ddot{x}$ as dv/dt, and sometimes you want to write it as $v\,dv/dx$ (see eq. (2.14)). Then you 'simply' separate variables and integrate. We'll go through the three special cases, and then we'll do some examples.

- *F is a function of t only: $F = F(t)$.*

  Since $a = d^2x/dt^2$, we simply have to integrate $F = ma$ twice to obtain $x(t)$. Let's do this in a very systematic way, just to get used to the general procedure. Write $F = ma$ as

  $$m\frac{dv}{dt} = F(t). \tag{2.11}$$

  Separate variables and integrate both sides to obtain

  $$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')dt'. \tag{2.12}$$

  (Primes have been put on the integration variables so that we don't confuse them with the limits of integration.) This yields v as a function of t, $v(t)$. Then separate variables in $dx/dt = v(t)$ and integrate to obtain

  $$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'. \tag{2.13}$$

  This yields x as a function of t, $x(t)$. This procedure may seem like a cumbersome way to simply integrate something twice. That's because it is. But the technique proves more useful in the following case.

[4] You can always solve for $x(t)$ *numerically*, to any desired accuracy. This is discussed in Appendix D.

- *F is a function of x only: $F = F(x)$.*

  Use
$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{2.14}$$
  to write $F = ma$ as
$$mv\frac{dv}{dx} = F(x). \tag{2.15}$$
  Separate variables and integrate both sides to obtain
$$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')dx'. \tag{2.16}$$
  The left side will contain the square of $v(x)$. Taking a square root, this gives v as a function of x, $v(x)$. Separate variables in $dx/dt = v(x)$ to obtain
$$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'. \tag{2.17}$$
  This gives t as a function of x, and hence (in principle) x as a function of t, $x(t)$. The unfortunate thing about this case is that the integral in eq. (2.17) might not be doable. And even if it is, it might not be possible to invert $t(x)$ to produce $x(t)$.

- *F is a function of v only: $F = F(v)$.*

  Write $F = ma$ as
$$m\frac{dv}{dt} = F(v). \tag{2.18}$$
  Separate variables and integrate both sides to obtain
$$m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt'. \tag{2.19}$$
  This yields t as a function of v, and hence (in principle) v as a function of t, $v(t)$. Integrate $dx/dt = v(t)$ to obtain $x(t)$ from
$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'. \tag{2.20}$$
  Note: If you want to find v as a function of x, $v(x)$, you should write a as $v(dv/dx)$ and integrate
$$m\int_{v_0}^{v(x)} \frac{v'\,dv'}{F(v')} = \int_{x_0}^{x} dx'. \tag{2.21}$$
  You may then obtain $x(t)$ from eq. (2.17), if desired.

When dealing with the initial conditions, we have chosen to put them in the limits of integration above. If you wish, you can perform the integrals without any limits, and just tack on a constant of integration to your result. The constant is then determined by the initial conditions.

Again, you do *not* have to memorize the above three procedures, because there are variations, depending on what you want to solve for. All you have to remember is that $\ddot{x}$ can be written as either dv/dt or $v\,dv/dx$. One of these will get the job done (the one that makes only two out of the three variables, t,x,v, appear in your differential equation). And then be prepared to separate variables and integrate as many times as needed.

> a is dv by dt.
> Is this useful? There's no guarantee.
> If it leads to "Oh, heck!"'s,
> Take dv by dx,
> And then write down its product with v.

Example 1 (Gravitational force): A particle of mass m is subject to a constant force $F = -mg$. The particle starts at rest at height h. Since this constant force falls into all of the above three categories, we should be able to solve for the motion in two ways.

(a) Find $y(t)$ by writing a as dv/dt.

(b) Find $y(t)$ by writing a as $v\,dv/dy$.

Solution:

(a) $F = ma$ gives $dv/dt = -g$. Integrating this yields $v = -gt + C$, where C is a constant of integration. The initial condition $v(0) = 0$ says that $C = 0$. Hence, $dy/dt = -gt$. Integrating this and using $y(0) = h$ gives

$$y = h - \frac{1}{2}gt^2. \tag{2.22}$$

(b) $F = ma$ gives $v\,dv/dy = -g$. Separating variables and integrating gives $v^2/2 = -gy + C$. The initial condition $v(0) = 0$ yields $v^2/2 = -gy + gh$. Therefore, $v \equiv dy/dt = -\sqrt{2g(h-y)}$ (we have chosen the negative square root, because the particle is falling). Separating variables then gives

$$\int \frac{dy}{\sqrt{h-y}} = -\sqrt{2g}\int dt. \tag{2.23}$$

This yields $2\sqrt{h-y} = \sqrt{2g}\,t$, where we have used the initial condition $y(0) = h$. Hence, $y = h - gt^2/2$, in agreement with part (a) (which was clearly the simpler method for this problem).

Example 2 (Dropped ball): A beach-ball is dropped from rest at height h. Assume[5] that the drag force from the air is $F_d = -\beta v$. Find the velocity and height as a function of time.

Solution: For simplicity in future formulas, let's write the drag force as $F_d = -\beta v \equiv -m\alpha v$. Taking upward to be the positive y direction, the force on the ball is then

$$F = -mg - m\alpha v. \tag{2.24}$$

(If we had chosen downward to be the positive direction, then the force would have been $mg - m\alpha v$.) Note that v is negative here, so the drag force points upward, as it should. Writing $F = m\,dv/dt$, and separating variables, gives

$$\int_0^{v(t)} \frac{dv'}{-g - \alpha v'} = \int_0^t dt'. \tag{2.25}$$

The integration yields $\ln(1 + \alpha v/g) = -\alpha t$. Exponentiation then gives

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{2.26}$$

Integrating $dy/dt \equiv v(t)$ to obtain $y(t)$ yields

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'. \tag{2.27}$$

Therefore,

$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{2.28}$$

REMARKS:

(a) Let's look at some limiting cases. If t is very small (more precisely, if $\alpha t \ll 1$), then we can use $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in t. You can show that eq. (2.26) gives $v(t) \approx -gt$ (as it should, since the drag force is negligible at the start). And eq. (2.28) gives $y(t) \approx h - gt^2/2$, as expected.

We may also look at large t. In this case, $e^{-\alpha t}$ is essentially 0, so eq. (2.26) gives $v(t) \approx -g/\alpha$. (This is the terminal velocity. Its value makes sense, because it is the velocity for which the force $-mg - m\alpha v$ vanishes.) And eq. (2.28) gives $y(t) \approx h - (g/\alpha)t + g/\alpha^2$. Apparently, for large t, g/α^2 is the distance our ball lags behind another ball which starts out already at the terminal velocity, g/α.

(b) The speed of the ball obtained in eq. (2.26) depends on α, which was defined in the coefficient of the drag force, $F_d = -m\alpha v$. We explicitly wrote the m here just to make all of our formulas look a little nicer, but it should *not* be inferred that the speed of the ball is independent of m. The coefficient $m\alpha$ depends (in some complicated way) on the cross-sectional area, A, of the ball. Therefore, $\alpha \propto A/m$. Two balls of the same size, one made of lead and one made of styrofoam, will have the same A but different m's. Hence, their α's will be different, and they will fall at different rates.

For heavy objects in a thin medium such as air, α is small, and so the drag effects are not very noticeable over short distances.[6] Massive objects fall at roughly the same rate. If the air were a bit thicker, different objects would fall at noticeably different rates, and maybe it would have taken Galileo a bit longer to come to his conclusions.

[5] The drag force is roughly proportional to v as long as the speed is fairly slow (up to, say, 50 m/s, but this depends on various things). For larger speeds, the drag force is roughly proportional to v^2.

[6] In such a scenario, we would more likely have $F \propto v^2$, but the general conclusion about small effects still holds.

What would you have thought, Galileo,
If instead you dropped cows and did say, "Oh!
To lessen the sound
Of the moos from the ground,
They should fall not through air, but through mayo!" ♣

2.4 Projectile motion

Consider a ball thrown through the air (not necessarily vertically). Let x and y be the horizontal and vertical positions, respectively. The force in the x-direction is $F_x = 0$, and the force in the y-direction is $F_y = -mg$. So $\mathbf{F} = m\mathbf{a}$ gives

$$\ddot{x} = 0, \qquad \text{and} \qquad \ddot{y} = -g. \tag{2.29}$$

Note that these two equations are "decoupled". That is, there is no mention of y in the equation for $\ddot{x}$, and vice-versa. The motions in the x- and y-directions are therefore completely independent.

If the initial position and velocity are (X, Y) and (V_x, V_y), then we can easily integrate eqs. (2.29) to obtain

$$\begin{aligned} \dot{x}(t) &= V_x, \\ \dot{y}(t) &= V_y - gt. \end{aligned} \tag{2.30}$$

Integrating again gives

$$\begin{aligned} x(t) &= X + V_x t, \\ y(t) &= Y + V_y t - \frac{1}{2}gt^2. \end{aligned} \tag{2.31}$$

These equations for the speeds and positions are all you need to solve a projectile problem. (Of course, we've neglected air resistance here. Things get a bit complicated when that is included.)

Example (Throwing a ball):

(a) For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum horizontal distance? Assume that the ground is level, and that the ball is released from ground level.

(b) What is the optimal angle if the ground is sloped upward at an angle β (or downward, if β is negative)?

Solution:

(a) Let the inclination angle be θ, and let the initial speed be v. Then the horizontal speed is (always) $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$.

Let d be the horizontal distance traveled, and let t be the time in the air. Then the vertical speed is zero at time $t/2$, so eq. (2.30) says that $v_y = g(t/2)$. Hence, $t = 2v_y/g$. (Alternatively, the time of flight can be found from eq. (2.31), which says that the ball returns to ground level when $v_y t = gt^2/2$.) Eq. (2.31) says that $d = v_x t$. Using $t = 2v_y/g$ in this gives

$$d = \frac{2v_x v_y}{g} = \frac{v^2}{g}(2\sin\theta\cos\theta) = \frac{v^2}{g}\sin 2\theta. \tag{2.32}$$

The $\sin 2\theta$ factor has a maximum at

$$\theta = \frac{\pi}{4}. \tag{2.33}$$

The maximum distance traveled is then $d = v^2/g$.

For $\theta = \pi/4$, you can show that the maximum height achieved is $v^2/4g$. This may be compared to the maximum height of $v^2/2g$ (as you can show) if the ball is thrown straight up.

Note that any possible distance you might wish to find must be proportional to v^2/g, by dimensional analysis. The only question is what the numerical factor is.

(b) If the ground is sloped at an angle β, then the equation for the line of the ground is

$$y = (\tan\beta)x. \tag{2.34}$$

The path of the ball is given in terms of t by

$$x = (v\cos\theta)t, \qquad \text{and} \qquad y = (v\sin\theta)t - \frac{1}{2}gt^2. \tag{2.35}$$

The t for which $y = (\tan\beta)x$ (that is, the place where that path of the ball intersects the line of the ground) may be solved for to obtain

$$t = \frac{2v}{g}(\sin\theta - \tan\beta\cos\theta). \tag{2.36}$$

(There is, of course, also the solution $t = 0$.) Plugging this into our expression for x in eq. (2.35) gives

$$x = \frac{2v^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta). \tag{2.37}$$

We must now maximize this value for x (which is the same as maximizing the distance along the slope). Taking the derivative with respect to θ gives (with the help of the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$) $\tan\beta = -\cot 2\theta \equiv -\tan(\pi/2 - 2\theta)$. Therefore, $\beta = -(\pi/2 - 2\theta)$, so we have

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right). \tag{2.38}$$

In other words, the throwing angle should bisect the angle between the ground and the vertical. For $\beta = \pi/2$, we have $\theta = \pi/2$, as it should be. For $\beta = 0$, we have $\theta = \pi/4$, as found in part (a). For $\beta = -\pi/2$, we have $\theta = 0$, which makes sense.

The classic demonstration of the independence of the x- and y-motions is the following. Fire a bullet horizontally (or, preferably, just imagine firing a bullet horizontally), and at the same time drop a bullet from the height of the gun. Which bullet will hit the ground first? (Neglect air resistance, and the curvature of the earth, etc.) The answer is that they will hit the ground at the same time, because the effect of gravity on the two y-motions is exactly the same, independent of what is going on in the x-direction.

2.5 Motion in a plane, polar coordinates

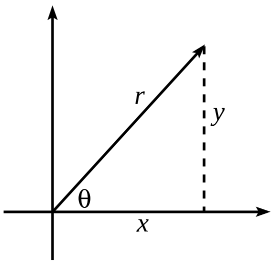

Figure 2.5

When dealing with problems where the motion lies in a plane, it is often convenient to work with polar coordinates, r and θ. These are related to the cartesian coordinates by (see Fig. 2.5)

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta. \tag{2.39}$$

The goal of the present section is to determine what $\mathbf{F} = m\mathbf{a} \equiv m\ddot{\mathbf{r}}$ looks like when written in terms of polar coordinates.

At a given position $\mathbf{r}$ in the plane, the basis vectors in polar coordinates are $\hat{\mathbf{r}}$, which is a unit vector pointing in the radial direction; and $\hat{\boldsymbol{\theta}}$, which is a unit vector pointing in the counterclockwise tangential direction. In polar coords, a general vector may therefore be written as

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{2.40}$$

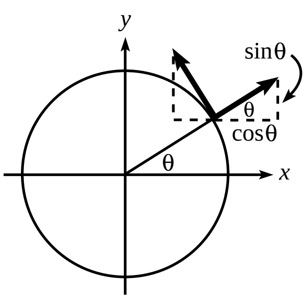

Figure 2.6

Note that the directions of the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors depend, of course, on $\mathbf{r}$.

The goal of this section is to find $\ddot{\mathbf{r}}$. So, in view of eq. (2.40), we must get a handle on the time derivative of $\hat{\mathbf{r}}$ (and we'll eventually need the derivative of $\hat{\boldsymbol{\theta}}$, also). In contrast with the cartesian basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), the polar basis vectors ($\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$) do indeed change as a point moves around in the plane. We may find $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ in the following way. In terms of the cartesian basis, Fig. 2.6 shows that

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}. \end{aligned} \tag{2.41}$$

Taking the time derivative of these equations gives

$$\begin{aligned} \dot{\hat{\mathbf{r}}} &= -\sin\theta\dot{\theta}\hat{\mathbf{x}} + \cos\theta\dot{\theta}\hat{\mathbf{y}}, \\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\dot{\theta}\hat{\mathbf{x}} - \sin\theta\dot{\theta}\hat{\mathbf{y}}. \end{aligned} \tag{2.42}$$

Using eqs. (2.41), we then arrive at the nice clean expressions,

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}, \qquad \text{and} \qquad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}. \tag{2.43}$$

These relations are fairly evident from viewing what happens to the basis vectors as $\mathbf{r}$ moves a tiny distance in the tangential direction.

We may now start differentiating eq. (2.40). One derivative gives

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \end{aligned} \tag{2.44}$$

This is quite clear, since $\dot{r}$ is the speed in the radial direction, and $r\dot{\theta}$ is the speed in the tangential direction (which is often written as ωr, where $\omega \equiv \dot{\theta}$ is the angular speed, or 'angular frequency').[7]

[7]For $r\dot{\theta}$ to be the tangential speed, we must of course measure θ in radians and not degrees. Then $r\theta$ is by definition the distance along the circumference; so $r\dot{\theta}$ is the speed along the circumference.

Differentiating eq. (2.44) then gives

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}}) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}}) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \end{aligned} \tag{2.45}$$

Finally, equating $m\ddot{\mathbf{r}}$ with $\mathbf{F} \equiv F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ gives the radial and tangential forces as

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2), \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \tag{2.46}$$

Exercise 7 gives a slightly different derivation of these equations.

Let's look at each of the four terms on the right-hand sides of eqs. (2.46).

- The $m\ddot{r}$ term is quite intuitive. For radial motion, it simply states that $F = ma$ along the radial direction.
- The $mr\ddot{\theta}$ term is also quite intuitive. For circular motion, it states that $F = ma$ along the tangential direction.
- The $-mr\dot{\theta}^2$ term is also fairly clear. For circular motion, it says that the radial force is $-m(r\dot{\theta})^2/r = -mv^2/r$, which is the familiar term that causes the centripetal acceleration.
- The $2m\dot{r}\dot{\theta}$ term is not so obvious. It is called the *Coriolis* force. There are various ways to look at this term. One is that it exists in order to keep the angular momentum conserved. We'll have much more to say about this in Chapter 9.

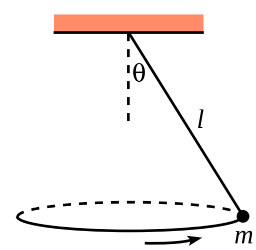

Figure 2.7

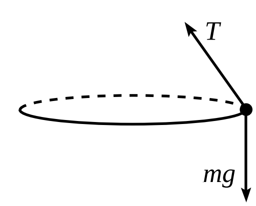

Figure 2.8

Example (Circular pendulum): A mass hangs from a string of length ℓ. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle of θ with the vertical (see Fig. 2.7). What is the angular frequency, ω, of this motion?

Solution: The mass travels in a circle, so the horizontal radial force is $F_r = mr\dot{\theta}^2 \equiv mr\omega^2$ (with $r = \ell\sin\theta$), directed radially inward. The forces on the mass are the tension in the string, T, and gravity, mg (see Fig. 2.8). There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions gives, respectively,

$$\begin{aligned} T\cos\theta &= mg, \\ T\sin\theta &= m(\ell\sin\theta)\omega^2. \end{aligned} \tag{2.47}$$

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell\cos\theta}}. \tag{2.48}$$

Note that if $\theta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which equals the frequency of a plane pendulum of length ℓ. And if $\theta \approx 90^\circ$, then $\omega \to \infty$, as it should.

2.6 Exercises

Section 2.2: Free-body diagrams

1. **A Peculiar Atwood's Machine**

   Consider an Atwood's machine (see Fig. 2.9) consisting of N masses, M, $M/2$, $M/4$, ..., $M/2^{N-1}$. (All the pulleys and strings are massless, as usual.)

   (a) Put a mass $M/2^{N-1}$ at the free end of the bottom string. What are the accelerations of all the masses?

   (b) Remove the mass $M/2^{N-1}$ (which was arbitrarily small, for very large N) that was attached in part (a). What are the accelerations of all the masses, now that you've removed this infinitesimal piece?

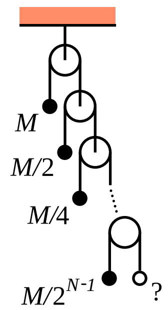

Figure 2.9

2. **Accelerated Cylinders** $**$

   Three identical cylinders are arranged in a triangle as shown in Fig. 2.10, with the bottom two lying on the ground. The ground and the cylinders are frictionless.

   You apply a constant horizontal force (directed to the right) on the left cylinder. Let a be the acceleration you give to the system. For what range of a will all three cylinders remain in contact with each other?

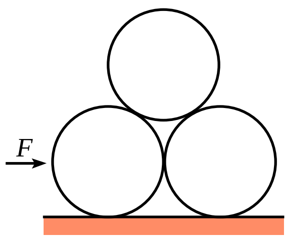

Figure 2.10

Section 2.3: Solving differential equations

3. $-bv^2$ **force** $*$

   A particle of mass m is subject to a force $F(v) = -bv^2$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

4. $-kx$ **force** $**$

   A particle of mass m is subject to a force $F(x) = -kx$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

5. kx **force** $**$

   A particle of mass m is subject to a force $F(x) = kx$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

Section 2.4: Projectile motion

6. **Newton's apple** $*$

   Newton is tired of apples falling on his head, so he decides to throw a rock at one of the larger and more formidable-looking apples positioned directly above his favorite sitting spot. Forgetting all about his work on gravitation (along with general common sense), he aims the rock directly at the apple (see Fig. 2.11). To his surprise, however, the apple falls from the tree just as he

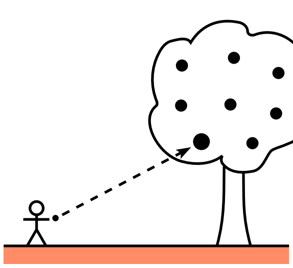

Figure 2.11

releases the rock. Show that the rock will hit the apple.[8]

Section 2.5: Motion in a plane, polar coordinates

7. **Derivation of F_r and F_θ** $**$

   In cartesian coords, a general vector takes the form

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ &= r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}. \end{aligned} \tag{2.49}$$

   Derive eqs. (2.46) by taking two derivatives of this expression for $\mathbf{r}$, and then using eqs. (2.41) to show that the result may be written in the form of eq. (2.45). (Unlike $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change with time.)

[8] This problem suggests a way in which William Tell and his son might survive their ordeal if they were plopped down on a planet with unknown gravitational constant (provided the son isn't too short or g isn't too big).

2.7 Problems

Section 2.2: Free-body diagrams

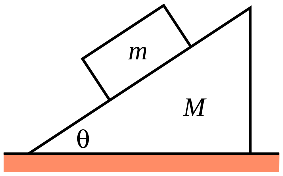

Figure 2.12

1. **Sliding plane** ***

   A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (see Fig. 2.12). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?

2. **Sliding down a plane** **

   (a) A block slides down a frictionless plane from the point $(0, y)$ to the point $(b, 0)$, where b is given. For what value of y does the journey take the shortest time? What is this time?

   (b) Answer the same questions in the case where there is a coefficient of kinetic friction, μ, between the block and the plane.

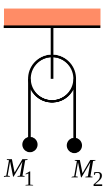

Figure 2.13

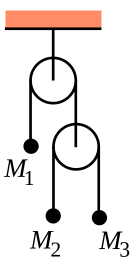

Figure 2.14

3. **Atwood's machine** **

   (a) A massless pulley hangs from a fixed support. A string connecting two masses, M_1 and M_2, hangs over the pulley (see Fig. 2.13). Find the accelerations of the masses.

   (b) Consider now the double-pulley system with masses M_1, M_2, and M_3 (see Fig. 2.14). Find the accelerations of the masses.

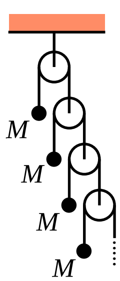

Figure 2.15

4. **Infinite Atwood's machine** ***

   Consider the infinite Atwood's machine shown in Fig. 2.15. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to M, and all the pulleys and strings are massless.

   The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?

   (You may define this infinite system as follows. Consider it to be made of N pulleys, with a non-zero mass replacing what would have been the $(N + 1)$st pulley. Then take the limit as $N \to \infty$. It is not necessary, however, to use this exact definition.)

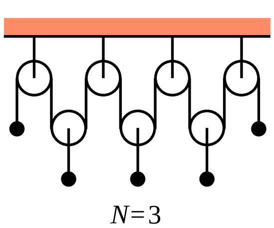

Figure 2.16

5. **Line of Pulleys** *

   $N + 2$ equal masses hang from a system of pulleys, as shown in Fig. 2.16. What is the acceleration of the masses at the end of the string?

6. **Ring of Pulleys** $**$

Consider the system of pulleys shown in Fig. 2.17. The string (which is a loop with no ends) hangs over N fixed pulleys. N masses, m_1, m_2, …, m_N, are attached to N pulleys which hang on the string. Find the acceleration of each mass.

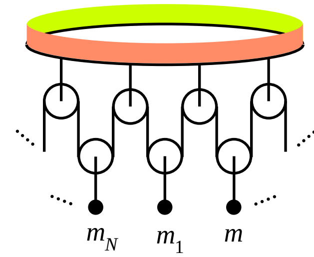

Figure 2.17

Section 2.3: Solving differential equations

7. **Exponential force**

A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are 0. Find $x(t)$.

8. **Falling chain** $**$

(a) A chain of length ℓ is held on a frictionless horizontal table, with a length y_0 hanging over the edge. The chain is released. As a function of time, find the length that hangs over the edge. (Don't bother with t after the chain loses contact with the table.) Also, find the speed of the chain right when it loses contact with the table.

(b) Do the same problem, but now let there be a coefficient of friction μ between the chain and the table. (Assume that the chain initially hangs far enough over the edge so that it will indeed fall when released.)

9. **Ball thrown upward** $***$

A beach-ball is thrown upward with initial speed v_0. Assume that the drag force is $F = -m\alpha v$. What is the speed of the ball, v_f, when it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

Section 2.4: Projectile motion

10. **Throwing a ball from a cliff** $**$

A ball is thrown from the edge of a cliff of height h. At what inclination angle should it be thrown so that it travels a maximum horizontal distance? Assume that the ground below the cliff is level.

11. **Redirected horizontal motion** $*$

A ball falls from height h. It bounces off a surface at height y (with no loss in speed). The surface is inclined at 45°, so that the ball bounces off horizontally. What should y be so that the ball travels a maximum horizontal distance?

12. **Redirected general motion** $*$

A ball falls from height h. It bounces off a surface at height y (with no loss in speed). The surface is inclined so that the ball bounces off at an angle of θ with respect to the horizontal. What should y and θ be so that the ball travels a maximum horizontal distance?

13. **Maximum length of trajectory** ***

A ball is thrown at speed v from zero height on level ground. Let θ_0 be the angle at which the ball should be thrown so that the distance traveled *through the air* is maximum. Show that θ_0 satisfies

$$1 = \sin\theta_0 \ln\left(\frac{1 + \sin\theta_0}{\cos\theta_0}\right).$$

(The solution is found numerically to be $\theta_0 \approx 56.5°$.)

14. **Maximum area under trajectory** *

A ball is thrown at speed v from zero height on level ground. At what angle should the ball be thrown so that the area under the trajectory is maximum?

15. **Bouncing ball** *

A ball is thrown straight upward so that it reaches a height h. It falls down and bounces repeatedly. After each bounce, it returns to a certain fraction f of its previous height. Find the total distance traveled, and also the total time, before it comes to rest. What is its average speed?

2.8 Solutions

1. **Sliding plane**

Let F be the normal force between the block and the plane. Then the various $F = ma$ equations (vertical and horizontal for the block, and horizontal for the plane) are

$$\begin{aligned} mg - F\cos\theta &= ma_y \\ F\sin\theta &= ma_x \\ F\sin\theta &= MA_x. \end{aligned} \tag{2.50}$$

(We've chosen positive a_x and a_y to be leftward and downward, respectively, and positive A_x to be rightward.)

There are four unknowns here (a_x,a_y,A_x,F), so we need one more equation. This last equation is the constraint that the block remains in contact with the plane. The horizontal distance between the block and its starting point on the plane is $(a_x + A_x)t^2/2$, and the vertical distance is $a_y t^2/2$. The ratio of these distances must equal $\tan\theta$ if the block is to remain on the plane. Therefore, we must have

$$\frac{a_y}{a_x + A_x} = \tan\theta. \tag{2.51}$$

Using eqs. (2.50), this becomes

$$\begin{aligned} \frac{g - \frac{F}{m}\cos\theta}{\frac{F}{m}\sin\theta + \frac{F}{M}\sin\theta} &= \tan\theta \\ \Longrightarrow \qquad F &= g\left(\sin\theta\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right) + \frac{\cos\theta}{m}\right)^{-1}. \end{aligned} \tag{2.52}$$

The third of eqs. (2.50) then yields A_x, which may be written as

$$A_x = \frac{F\sin\theta}{M} = \frac{mg\tan\theta}{M(1+\tan^2\theta) + m\tan^2\theta}. \tag{2.53}$$

REMARKS: For given M and m, the angle θ_0 which maximizes A_x is found to be

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}. \tag{2.54}$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, we have $A_x \approx g/\tan\theta$. This makes sense, because m falls essentially straight down, and the plane gets squeezed out to the right.

In the limit $M \gg m$, we have $A_x \approx g(m/M)\tan\theta/(1+\tan^2\theta) = g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $a_x = (M/m)A_x \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of a_x implies that the acceleration of m *along* the plane is essentially equal to $a_x/\cos\theta \approx g\sin\theta$, as expected. ♣

2. **Sliding down a plane**

   (a) Let θ be the angle the plane makes with the horizontal. The component of gravity along the plane in $g\sin\theta$. The acceleration in the horizontal direction is then $a_x = g\sin\theta\cos\theta$. Since the horizontal distance is fixed, we simply want to maximize a_x. So $\theta = \pi/4$, and hence $y = b$.

   The time is obtained from $a_x t^2/2 = b$, with $a_x = g\sin\theta\cos\theta = g/2$. Therefore, $t = 2\sqrt{b/g}$.

(b) The normal force on the plane is $Mg\cos\theta$, so the friction force is $\mu Mg\cos\theta$. The acceleration along the plane is therefore $g(\sin\theta - \mu\cos\theta)$, and so the acceleration in the horizontal direction is $a_x = g(\sin\theta - \mu\cos\theta)\cos\theta$. We want to maximize this. Setting the derivative equal to zero gives

$$(\cos^2\theta - \sin^2\theta) + 2\mu\sin\theta\cos\theta = 0, \qquad \Longrightarrow \qquad \tan 2\theta = \frac{-1}{\mu}. \tag{2.55}$$

For $\mu \to 0$, this reduces to the answer in part (a). For $\mu \to \infty$, we obtain $\theta \approx \pi/2$, which makes sense.

To find t, we need to find a_x. Using $\sin 2\theta = 1/\sqrt{1+\mu^2}$ and $\cos 2\theta = -\mu/\sqrt{1+\mu^2}$, we have

$$\begin{aligned} a_x &= g\sin\theta\cos\theta - \mu g\cos^2\theta \\ &= \frac{g\sin 2\theta}{2} - \frac{\mu g(1+\cos 2\theta)}{2} \\ &= \frac{g}{2}(\sqrt{1+\mu^2} - \mu). \end{aligned} \tag{2.56}$$

(For $\mu \to \infty$, this behaves like $a_x \approx g/(4\mu)$.) Therefore, $a_x t^2/2 = b$ gives

$$\begin{aligned} t = \sqrt{\frac{2b}{a_x}} &= \frac{2\sqrt{b/g}}{\sqrt{\sqrt{1+\mu^2} - \mu}} \\ &= 2\sqrt{b/g}\sqrt{\sqrt{1+\mu^2} + \mu}. \end{aligned} \tag{2.57}$$

(For $\mu \to \infty$, this behaves like $t \approx 2\sqrt{2\mu b/g}$.)

3. **Atwood's machine**

(a) Let T be the tension in the string. Let a be the acceleration of M_2 (with downward taken to be positive). Then $-a$ is the acceleration of M_1. So we have

$$\begin{aligned} M_1 g - T &= M_1(-a), \\ M_2 g - T &= M_2 a. \end{aligned} \tag{2.58}$$

Subtracting the two equations yields

$$a = g\frac{M_2 - M_1}{M_1 + M_2}. \tag{2.59}$$

As a double-check, this has the correct limits when $M_2 \gg M_1$, $M_2 \ll M_1$, and $M_2 = M_1$, namely $a \approx g$, $a \approx -g$, and $a = 0$, respectively.

We may also solve for the tension, $T = 2M_1M_2/(M_1 + M_2)$. If $M_1 = M_2 \equiv M$, then $T = Mg$, as it should. If $M_1 \ll M_2$, then $T \approx 2M_1 g$, as it should (because then the net upward force on M_1 is $M_1 g$, so its acceleration equals g upwards, as it must, since M_2 is essentially in free-fall).

(b) The key here is that since the pulleys are massless, there can be no net force on them, so the tension in the bottom string must be half of that in the top string. Let these be $T/2$ and T, respectively. Let a_p be the acceleration of the bottom pulley, and let a be the acceleration of M_3 relative to the bottom pulley (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - T &= M_1(-a_p), \\ M_2 g - \frac{T}{2} &= M_2(a_p - a), \\ M_3 g - \frac{T}{2} &= M_3(a_p + a). \end{aligned} \tag{2.60}$$

Solving for a_p and a gives

$$a_p = g\frac{4M_2M_3 - M_1(M_2+M_3)}{4M_2M_3 + M_1(M_2+M_3)}, \qquad a = g\frac{2M_1(M_3 - M_2)}{4M_2M_3 + M_1(M_2+M_3)}. \tag{2.61}$$

The accelerations of M_2 and M_3, namely $a_p - a$ and $a_p + a$, are

$$\begin{aligned} a_p - a &= g\frac{4M_2M_3 + M_1(M_2 - 3M_3)}{4M_2M_3 + M_1(M_2+M_3)}, \\ a_p + a &= g\frac{4M_2M_3 + M_1(M_3 - 3M_2)}{4M_2M_3 + M_1(M_2+M_3)}. \end{aligned} \tag{2.62}$$

You can check various limits. One nice one is where M_2 is much less than both M_1 and M_3. The accelerations of M_1, M_2, and M_3 are then

$$-a_p = g, \qquad a_p - a = -3g, \qquad a_p + a = g, \tag{2.63}$$

(with downward taken to be positive).

4. **Infinite Atwood's machine**

First Solution: Consider the following auxiliary problem.

Problem: Two set-ups are shown in Fig. 2.18. The first contains a hanging mass m. The second contains a hanging pulley, over which two masses, M_1 and M_2, hang. Let both supports have acceleration a_s downward. What should m be, in terms of M_1 and M_2, so that the tension in the top string is the same in both cases?

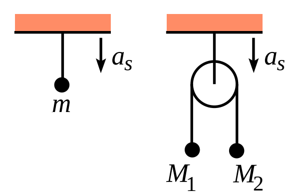

Figure 2.18

Answer: In the first case, we have

$$mg - T = ma_s. \tag{2.64}$$

In the second case, let a be the acceleration of M_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - \frac{T}{2} &= M_1(a_s - a), \\ M_2 g - \frac{T}{2} &= M_2(a_s + a). \end{aligned} \tag{2.65}$$

Note that if we define $g' \equiv g - a_s$, then we may write these three equations as

$$\begin{aligned} mg' &= T, \\ M_1 g' &= \frac{T}{2} - M_1 a, \\ M_2 g' &= \frac{T}{2} + M_2 a. \end{aligned} \tag{2.66}$$

The last two give $4M_1M_2g' = (M_1 + M_2)T$. The first equation then gives

$$m = \frac{4M_1M_2}{M_1 + M_2}. \tag{2.67}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This problem shows that the two-mass system in the second case may be equivalently treated as a mass m, as far as the upper string is concerned. ■

Now let's look at our infinite Atwood machine. Start at the bottom. (Assume that the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the above problem shows that the bottom two masses, M and x, may be treated as an effective mass $f(x)$, where

$$f(x) = \frac{4x}{1 + (x/M)}. \tag{2.68}$$

We may then treat the combination of the mass $f(x)$ and the next M as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass M and a mass $f^{(N-1)}(x)$ hanging over the top pulley.

We must determine the behavior of $f^N(x)$, as $N \to \infty$. The behavior is obvious by looking at a plot of $f(x)$ (which we'll let you draw). (Note that $x = 3M$ is a fixed point of f, i.e., $f(3M) = 3M$.) It is clear that no matter what x we start with, the iterations approach $3M$ (unless, of course, $x = 0$). So our infinite Atwood machine is equivalent to (as far as the top mass is concerned) just the two masses M and $3M$.

We then easily find that the acceleration of the top mass equals (net downward force)/(total mass) $= 2Mg/(4M) = g/2$.

As far as the support is concerned, the whole apparatus is equivalent to a mass $3M$. So $3Mg$ is the weight the support holds up.

Second Solution: If the gravity in the world were multiplied by a factor η, then the tension in all the strings would likewise be multiplied by η. (The only way to make a tension, i.e., a force, is to multiply a mass times g.) Conversely, if we put the apparatus on another planet and discover that all the tensions are multiplied by η, then we know the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (since the pulleys are massless). Let the acceleration of the second pulley be a_{p2}. Then the second pulley effectively lives in a world where the gravity is $g - a_{p2}$. If we imagine holding the string above the second pulley and accelerating downward at a_{p2} (so that our hand is at the origin of the new world), then we really haven't changed anything, so the tension in this string in the new world is still $T/2$.

But in this infinite setup, the system of all the pulleys except the top one is the same as the original system of all the pulleys. Therefore, by the arguments in the first paragraph, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_{p2}}. \tag{2.69}$$

Hence, $a_{p2} = g/2$. (Likewise, the relative acceleration of the second and third pulleys is $g/4$, etc.) But a_{p2} is also the acceleration of the top mass. So our answer is $g/2$.

Note that $T = 0$ also makes eq. (2.69) true. But this corresponds to putting a mass of zero at the end of a finite pulley system.

5. **Line of Pulleys**

Let m be the common mass, and let T be the tension in the string. Let a be the acceleration of the end masses, and let a' be the acceleration of the other masses (with downward taken to be positive). Then we have

$$\begin{aligned} T - mg &= ma, \\ 2T - mg &= ma'. \end{aligned} \tag{2.70}$$

The string has fixed length, therefore

$$N(2a') + a + a = 0. \tag{2.71}$$

Eliminating T from eqs. (2.70) gives $a' = 2a + g$. Combining this with eq. (2.71) then gives

$$a = \frac{-g}{2 + \frac{1}{N}}. \tag{2.72}$$

For $N = 0$ we have $a = 0$. For $N = 1$ we have $a = -g/3$. For larger N, a increases in magnitude until it equals $-g/2$ for $N \to \infty$.

6. **Ring of Pulleys**

Let T be the tension in the string. Then $F = ma$ for m_i gives

$$2T - m_i g = m_i a_i, \tag{2.73}$$

with upward taken to be positive.

But the string has a fixed length. Therefore, the sum of all the displacements of the masses is zero. Hence,

$$a_1 + a_2 + \cdots + a_N = 0. \tag{2.74}$$

If we divide eq. (2.73) by m_i, and then add the N such equations together, we then obtain

$$2T\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right) - Ng = 0. \tag{2.75}$$

Substituting this value for T into (2.73) gives

$$a_i = g\left(\frac{N}{m_i\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right)} - 1\right). \tag{2.76}$$

If all the masses are equal, then all $a_i = 0$. If $m_k = 0$ (and all the others are not 0), then $a_k = (N-1)g$, and all the other $a_i = -g$.

7. **Exponential force**

We are given $\ddot{x} = e^{-bt}$. Integrating this w.r.t. time gives $v(t) = -e^{-bt}/b + A$. Integrating again gives $x(t) = e^{-bt}/b^2 + At + B$. The initial conditions are $0 = v(0) = -1/b + A$ and $0 = x(0) = 1/b^2 + B$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}. \tag{2.77}$$

For $t \to \infty$, the speed is $v \to 1/b$. The particle eventually lags a distance $1/b^2$ behind a particle that starts at the same position but with speed $v = 1/b$.

8. **Falling chain**

(a) Let $y(t)$ be the length hanging over the edge at time t. Let the density of the chain be ρ. Then the total mass is $M = \rho\ell$, and the mass hanging over the edge is ρy. The downward force on the chain (that isn't countered by a normal force from the table) is therefore $(\rho y)g$, so $F = ma$ gives

$$\rho g y = \rho \ell \ddot{y} \qquad \Longrightarrow \qquad \ddot{y} = \frac{g}{\ell} y. \tag{2.78}$$

The solution to this equation is

$$y(t) = Ae^{\alpha t} + Be^{-\alpha t}, \qquad \text{where } \alpha \equiv \sqrt{\frac{g}{\ell}}. \tag{2.79}$$

Taking the derivative of this to obtain $\dot{y}(t)$, and using the given information that $\dot{y}(0) = 0$, we find $A = B$. Using $y(0) = y_0$, we then find $A = B = y_0/2$. So the length that hangs over the edge is

$$y(t) = \frac{y_0}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) \equiv y_0 \cosh(\alpha t). \tag{2.80}$$

And the speed is

$$\dot{y}(t) = \frac{\alpha y_0}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) \equiv \alpha y_0 \sinh(\alpha t). \tag{2.81}$$

The time T that satisfies $y(T) = \ell$ is given by $\ell = y_0 \cosh(\alpha T)$. Using $\sinh x = \sqrt{\cosh^2 x - 1}$, we find that the speed of the chain right when it loses contact with the table is

$$\dot{y}(T) = \alpha y_0 \sinh(\alpha T) = \alpha\sqrt{\ell^2 - y_0^2} \equiv \sqrt{g\ell}\sqrt{1 - \eta_0^2}, \tag{2.82}$$

where $\eta_0 \equiv y_0/\ell$ is the initial fraction hanging over the edge. If $\eta_0 \approx 0$, then the speed at T is $\sqrt{g\ell}$ (which is clear, since the center-of-mass falls a distance $\ell/2$) Also, you can show that T goes to infinity logarithmically as $\eta_0 \to 0$.

(b) The normal force on the table is $g\rho(\ell - y)$, so the friction force opposing gravity is $\mu g \rho(\ell - y)$. Therefore, $F = ma$ gives

$$\rho g y - \mu g \rho(\ell - y) = \rho \ell \ddot{y}. \tag{2.83}$$

This equation is valid only if the gravitational force is greater than the friction force (i.e., the left-hand side is positive), otherwise the chain just sits there. The left-hand side is positive if $y > \mu\ell/(1+\mu)$. Let us define a new variable $z \equiv y - \mu\ell/(1+\mu)$. (So our ending point, $y = \ell$, corresponds to $z = \ell/(1+\mu)$.) Then eq. (2.83) becomes

$$\ddot{z} = z\frac{g}{\ell}(1+\mu). \tag{2.84}$$

At this point, we can either repeat all the steps in part (a), with slightly different variables, or we can just realize that we now have the exact came problem, with the only change being that ℓ has turned into $\ell/(1+\mu)$. So we have

$$\begin{aligned} z(t) &= z_0 \cosh(\alpha' t), \qquad \text{where } \alpha' \equiv \sqrt{\frac{g(1+\mu)}{\ell}}, \\ \Longrightarrow \quad y(t) &= \left(y_0 - \frac{\mu\ell}{1+\mu}\right)\cosh(\alpha' t) + \frac{\mu\ell}{1+\mu}. \end{aligned} \tag{2.85}$$

And the final speed is

$$\dot{y}(T') = \dot{z}(T') = \alpha' z_0 \sinh(\alpha T') = \alpha' \sqrt{\frac{\ell^2}{(1+\mu)^2} - z_0^2} \equiv \sqrt{\frac{g\ell}{1+\mu}}\sqrt{1-\eta_0'^2}, \tag{2.86}$$

where $\eta_0' \equiv z_0/[\ell/(1+\mu)]$ is the initial 'excess fraction'. That is, it is the excess length above the minimal length, $\mu\ell/(1+\mu)$, divided by the maximum possible excess length, $\ell/(1+\mu)$. If $\eta_0' \approx 0$, then the speed at T' is $\sqrt{g\ell/(1+\mu)}$.

9. **Ball thrown upward**

Let's take upward to be the positive direction. Then on the way up, the force is $F = -mg - m\alpha v$. (v is positive here, so the drag force points downward, as it should.)

The first thing we must do is find the maximum height, h, the ball reaches. You can use the technique in eqs. (2.19) and (2.20) to solve for $v(t)$ and then $y(t)$. But it is much simpler to use eq. (2.21) to solve for $v(y)$, and to then take advantage of the fact that we know the speed of the ball at the top, namely zero. Eq. (2.21) gives

$$\int_{v_0}^{0} \frac{v\,dv}{g+\alpha v} = -\int_0^h dy. \tag{2.87}$$

Write $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$ and integrate to obtain

$$\frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right) = h. \tag{2.88}$$

On the way down, the force is again $F = -mg - m\alpha v$. (v is negative here, so the drag force points upward, as it should.) If v_f is the final speed (we'll take v_f to be a positive number, so that the final velocity is $-v_f$), then eq. (2.21) gives

$$\int_0^{-v_f} \frac{v\,dv}{g+\alpha v} = -\int_h^0 dy. \tag{2.89}$$

This gives

$$-\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right) = h. \tag{2.90}$$

(This is the same as eq. (2.88), with v_0 replaced by $-v_f$.)

Equating the expressions for h in eqs. (2.88) and (2.90) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_f}\right). \tag{2.91}$$

Remarks: Let's find approximate values for h in eqs. (2.88) and (2.90), in the limit of small α (which is the same as large g). More precisely, let's look at the limit $\alpha v_0/g \ll 1$. Using $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$, we find

$$h \approx \frac{v_0^2}{2g} - \frac{\alpha v_0^3}{3g^2} \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g} + \frac{\alpha v_f^3}{3g^2}. \tag{2.92}$$

The leading terms, $v^2/2g$, are as expected.

For small α, we may find v_f in terms of v_0. Equating the two expressions for h in eq. (2.92) (and using the fact that $v_f \approx v_0$) gives

$$\begin{aligned} v_0^2 - v_f^2 &\approx \frac{2\alpha}{3g}(v_0^3 + v_f^3) \\ \Longrightarrow (v_0 + v_f)(v_0 - v_f) &\approx \frac{2\alpha}{3g}(v_0^3 + v_f^3) \\ \Longrightarrow 2v_0(v_0 - v_f) &\approx \frac{4\alpha}{3g}v_0^3 \\ \Longrightarrow v_0 - v_f &\approx \frac{2\alpha}{3g}v_0^2 \\ \Longrightarrow v_f &\approx v_0 - \frac{2\alpha}{3g}v_0^2. \end{aligned} \tag{2.93}$$

We may also make approximations for large α (or small g). In this case, eq. (2.88) gives $h \approx v_0/\alpha$. And eq. (2.90) gives $v_f \approx g/\alpha$ (because the log term must be a very large negative number, in order to yield a positive h). There is no way to relate v_f and h is this case, because the ball quickly reaches the terminal velocity of g/α, independent of h. ♣

Let's now find the time it takes for the ball to go up and to go down. If T_1 is the time of the upward path, then integrating eq. (2.19), with $F = -mg - m\alpha v$, from the start to the apex gives

$$T_1 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{2.94}$$

Likewise, the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{2.95}$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right). \tag{2.96}$$

Using eq. (2.91), we have

$$T_1 + T_2 = \frac{v_0 + v_f}{g}. \tag{2.97}$$

This is shorter than the time in vacuum, namely $2v_0/g$, because $v_f < v_0$.

REMARKS: The fact that the time here is shorter than the time in vacuum isn't all that obvious. On one hand, the ball doesn't travel as far in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any α, you can easily use eq. (2.94) to show that $T_1 < v_0/g$. T_2 is harder to get a handle on, since it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum.

The very simple form of eq. (2.97) suggests that there should be a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (2.97). This procedure only worked, of course, because the drag force was proportional to v. ♣

10. **Throwing a ball from a cliff**

Let the angle be θ, and let the speed be v. Then the horizontal speed is $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$.

The time it takes for the ball to hit the ground is given by $(v\sin\theta)t - gt^2/2 = -h$. Therefore,

$$t = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \alpha}\right), \qquad \text{where } \alpha \equiv \frac{2gh}{v^2}. \tag{2.98}$$

(The other solution for t corresponds to the ball being thrown backwards down through the cliff.) The horizontal distance traveled is $d = (v\cos\theta)t$, so

$$d = \frac{v^2}{g}\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + \alpha}\right). \tag{2.99}$$

We must maximize this function $d(\theta)$. Taking the derivative, multiplying through by $\sqrt{\sin^2\theta + \alpha}$, and setting the result equal to zero, gives

$$(\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\theta + \alpha} = \sin\theta\big(\alpha - (\cos^2\theta - \sin^2\theta)\big). \tag{2.100}$$

Squaring, simplifying, and using $\cos^2\theta = 1 - \sin^2\theta$, gives

$$\sin\theta = \frac{1}{\sqrt{2+\alpha}} \equiv \frac{1}{\sqrt{2 + 2gh/v^2}}. \tag{2.101}$$

This is the optimal angle. Plugging this into eq. (2.99) gives a maximum distance of

$$d_{\text{max}} = \frac{v^2}{g}\sqrt{1+\alpha} \equiv \frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}}. \tag{2.102}$$

If $h = 0$, then we obtain the result of the example in Section 2.4. If $h \to \infty$ or $v \to 0$, then $\theta \approx 0$, which makes sense.

11. **Redirected horizontal motion**

First Solution: Let v be the speed right after the bounce (which is the same as the speed right before the bounce). Integrating $mv\,dv/dy = -mg$ gives $mv^2/2 = mg(h-y)$ (where the constant of integration has been chosen so that $v = 0$ when $y = h$). This is simply the conservation-of-energy statement. So we have

$$v = \sqrt{2g(h-y)}. \tag{2.103}$$

The vertical speed is zero right after the bounce, so the time it takes to hit the ground is given by $gt^2/2 = y$. Hence $t = \sqrt{2y/g}$. So the horizontal distance, d, traveled is

$$d = vt = 2\sqrt{y(h-y)}. \tag{2.104}$$

Taking a derivative, we see that this function of y is maximum at

$$y = \frac{h}{2}. \tag{2.105}$$

The corresponding value of d is $d_{\text{max}} = h$.

Second Solution: Assume that the greatest distance, d_0, is obtained when $y = y_0$ (and let the speed at y_0 be v_0).

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling horizontally. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy, which will be introduced in Chapter 4, but which you're all familiar with anyway), so it will travel a horizontal distance d_0 from this point. The total horizontal distance traveled is therefore $2d_0$.

So to maximize d_0, we simply have to maximize the horizontal distance in this new situation. From the example in Section 2.4, we want the ball to leave the ground at a $45°$ angle. Since it leaves the ground with speed $\sqrt{2gh}$, one can easily show that such a ball will be traveling horizontally at a height $y = h/2$, and it will travel a distance $2d_0 = 2h$. Hence, $y_0 = h/2$, and $d_0 = h$.

12. **Redirected general motion**

First Solution: We will use the results of Problem 10, namely eqs. (2.102) and (2.101), which say that an object projected from a height y at speed v travels a maximum distance of

$$d = \frac{v^2}{g}\sqrt{1 + \frac{2gy}{v^2}}, \tag{2.106}$$

and the optimal angle yielding this distance is

$$\sin\theta = \frac{1}{\sqrt{2 + 2gh/v^2}}. \tag{2.107}$$

In the problem at hand, the object is dropped from a height h, so conservation of energy (or integration of $mv\,dv/dy = -mg$) says that the speed at height y is

$$v = \sqrt{2g(h - y)}. \tag{2.108}$$

Plugging this into eq. (2.106) shows that the maximum horizontal distance, as a function of y, is

$$d_{\max}(y) = 2\sqrt{h(h - y)}. \tag{2.109}$$

This is clearly maximum at $y = 0$, in which case the distance is $d_{\max} = 2h$. Eq. (2.107) then gives the associated optimal angle as $\theta = 45°$.

Second Solution: Assume that the greatest distance, d_0, is obtained when $y = y_0 \neq 0$ and $\theta = \theta_0$ (and let the speed at y_0 be v_0). We will show that this cannot be the case. We will do this by explicitly constructing a situation yielding a greater distance.

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling at an angle θ_0 with respect to the horizontal. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy), so it will travel a horizontal distance d_0 from this point. But the ball traveled a nonzero horizontal distance on its way up to the height y_0. We have therefore constructed a situation yielding a distance greater than d_0. Hence, the optimal setup cannot have $y_0 \neq 0$. Therefore, the maximum distance must be obtained when $y = 0$ (in which case the example in Section 2.4 says that the optimal angle is $\theta = 45°$).

If you want the ball to go even further, simply dig a (wide enough) hole in the ground and have the ball bounce from the bottom of the hole.

13. **Maximum length of trajectory**

The coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. Eliminating t gives

$$y = (\tan\theta)x - \frac{gx^2}{2v^2\cos^2\theta}. \qquad (2.110)$$

The length of the arc is twice the length up to the maximum. The maximum occurs at $t = (v/g)\sin\theta$, and hence $x = (v^2/g)\sin\theta\cos\theta$. So the length of the arc is

$$\begin{aligned} L &= 2\int_0^{(v^2/g)\sin\theta\cos\theta} \sqrt{1 + (dy/dx)^2}\, dx \\ &= 2\int_0^{(v^2/g)\sin\theta\cos\theta} \sqrt{1 + (\tan\theta - gx/v^2\cos^2\theta)^2}\, dx. \qquad (2.111) \end{aligned}$$

Letting $z \equiv \tan\theta - gx/v^2\cos^2\theta$, we obtain

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\int_0^{\tan\theta} \sqrt{1+z^2}\, dz \\ &= \frac{2v^2\cos^2\theta}{g}\frac{1}{2}\left(z\sqrt{1+z^2} + \ln\left(z + \sqrt{1+z^2}\,\right)\right)\Bigg|_0^{\tan\theta} \\ &= \frac{v^2}{g}\left(\sin\theta + \cos^2\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\right). \qquad (2.112) \end{aligned}$$

(As a double-check, some special cases are $L = 0$ at $\theta = 0$, and $L = v^2/g$ at $\theta = 90°$, as one can explicitly verify.) Taking the derivative to find the maximum, we have

$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\frac{\cos\theta}{1+\sin\theta}\,\frac{\cos^2\theta + (1+\sin\theta)\sin\theta}{\cos^2\theta} \qquad (2.113)$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right), \qquad (2.114)$$

as was to be shown.

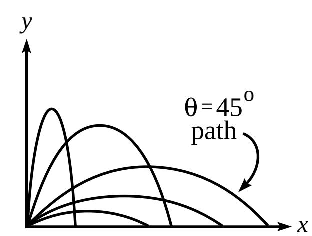

Figure 2.19

REMARK: The possible trajectories are shown in Fig. 2.19. Since it is well-known that $\theta = 45°$ provides the maximum *horizontal* distance, it is clear from the figure that the θ_0 yielding the arc of maximum length must satisfy $\theta_0 \geq 45°$. The exact angle, however, requires a detailed calculation. ♣

14. **Maximum area under trajectory**

The coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The time in the air is $T = 2(v\sin\theta)/g$. The area under the trajectory is

$$\begin{aligned} A &= \int_0^{x_{\max}} y\, dx \\ &= \int_0^{2v\sin\theta/g} \Big((v\sin\theta)t - gt^2/2\Big)v\cos\theta\, dt \\ &= \frac{2v^4}{3g^2}\sin^3\theta\cos\theta. \qquad (2.115) \end{aligned}$$

Taking a derivative, we find that the maximum occurs when $\tan\theta = \sqrt{3}$, i.e.,

$$\theta = 60^\circ. \tag{2.116}$$

The maximum area is then $A_{\text{max}} = \sqrt{3}v^4/8g^2$. (By dimensional analysis, we know that it has to be proportional to v^4/g^2.)

15. **Bouncing ball**

The ball travels $2h$ during the first up-and-down journey. It travels $2hf$ during the second, then $2hf^2$ during the third, and so on. Therefore, the total distance traveled is

$$\begin{aligned} D &= 2h(1 + f + f^2 + f^3 + \cdots) \\ &= \frac{2h}{1-f}. \end{aligned} \tag{2.117}$$

The time it takes to fall down during the first up-and-down is obtained from $h = gt^2/2$. So the time for the first up-and-down is $2t = 2\sqrt{2h/g}$. The time for the second up-and-down will likewise be $2\sqrt{2(hf)/g}$. Each successive time decreases by a factor of $\sqrt{f}$. The total time is therefore

$$\begin{aligned} T &= 2\sqrt{\frac{2h}{g}}(1 + f^{1/2} + f^1 + f^{3/2} + \cdots) \\ &= 2\sqrt{\frac{2h}{g}}\frac{1}{1-\sqrt{f}}. \end{aligned} \tag{2.118}$$

(Note that if f is exactly equal to 1, then the summations of the above series' are not valid.)

The average speed

$$\frac{D}{T} = \frac{\sqrt{gh/2}}{1+\sqrt{f}}. \tag{2.119}$$

REMARK: For $f \approx 1$, the average speed is roughly half of the average speed for $f \approx 0$. This may seem somewhat counterintuitive, because in the $f \approx 0$ case the ball slows down far sooner than in the $f \approx 1$ case. But the point is that the $f \approx 0$ case consists of essentially only one bounce, and the average speed for that bounce is the largest of any bounce. Both D and T are smaller for $f \approx 0$ than for $f \approx 1$; but T is smaller by a larger factor. ♣